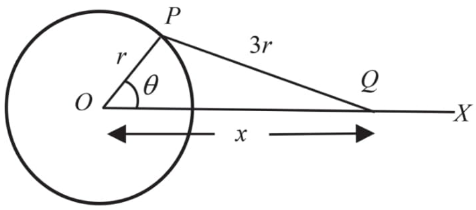
1 The curve with equation $y^2 = x^2 + 9$ is transformed by a stretch with scale factor 2 parallel to the x-axis, followed by a translation of 4 units in the negative x-direction, followed by a translation of $\frac{1}{2}$ units in the positive y-direction.

Find the equation of the new curve and state the equations of any asymptote(s). Sketch the new curve, indicating the coordinates of any turning points. [6]

2 The diagram shows a mechanism for converting rotational motion into linear motion. The point P, on the circumference of a disc of radius r, rotates about a fixed point O. The point Q moves along the line OX, and P and Q are connected by a rod of fixed length $3r$. As the disc rotates, the point Q is made to slide backwards and forwards along OX. At time t, angle POQ is θ, measured anticlockwise from OX, and the distance OQ is x.



(i) Show that $x = r\left(\cos\theta + \sqrt{9-\sin^2\theta}\right)$. [2]

(ii) State the maximum value of x. [1]

(iii) Express x as a polynomial in θ if θ is sufficiently small for θ^3 and higher powers of θ are to be neglected. [3]

3 Without using a calculator, solve the inequality $\dfrac{x^2-3x+4}{x+2} \geq 2x+1$. Hence solve the inequality $\dfrac{a^{2x}+3a^x+4}{a^x-2} \leq 2a^x - 1$ where $a > 2$. [6]

4 (i) Using double angle formula, prove that $\sin^4\theta = \frac{1}{8}\left(3-4\cos 2\theta + \cos 4\theta\right)$. [2]

(ii) By using the substitution $x = 2\cos\theta$, find the exact value of $\int_0^2 \left(4-x^2\right)^{\frac{3}{2}}\,dx$. [4]

H

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

Candidate Name		
CT Class	1 8	Centre Number/ Index Number: /

MATHEMATICS 9758/01

Paper 1

2nd September 2019

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	**Mark**
1	
2	
3	
4	
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6	
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10	
11	
12	
Total	

This document consists of **6** printed pages.

5 Relative to the origin O, the points A, B, and C, have non-zero position vectors $\mathbf{a}$, $\mathbf{b}$, and $3\mathbf{a}$ respectively. D lies on AB such that $AD = \lambda AB$, where $0 < \lambda < 1$.

(i) Write down a vector equation of the line OD. [1]

(ii) The point E is the midpoint of BC. Find the value of λ if E lies on the line OD. Show that the area of ΔBED is given by $k\left|\mathbf{a}\times\mathbf{b}\right|$, where k is a constant to be determined. [5]

6 The function f is given by $\mathrm{f}: x \mapsto 2x^2+4x+k$ for $-5 \le x < a$, where a and k are constants and $k > 2$.

(i) State the largest value of a for the inverse of f to exist. [1]

For the value of a found in **(i)**,

(ii) find $\mathrm{f}^{-1}(x)$ and the domain of f^{-1}, leaving your answer in terms of k, [3]

(iii) on the same diagram, sketch the graphs of $y = \mathrm{ff}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$, labelling your graphs clearly. Determine the number of solutions to $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [4]

7 A spherical tank with negligible thickness and internal radius a cm contains water. At time t s, the water surface is at a height x cm above the lowest point of the tank and the volume of water in the tank, V cm^3, is given by $V = \frac{1}{3}\pi x^2(3a-x)$. Water flows from the tank, through an outlet at its lowest point, at a rate $\pi k\sqrt{x}$ cm^3 s^{-1}, where k is a positive constant.

(i) Show that $\left(2ax - x^2\right)\frac{\mathrm{d}x}{\mathrm{d}t} = -k\sqrt{x}$. [2]

(ii) Find the general solution for t in terms of x, a and k. [3]

(iii) Find the ratio $T_1 : T_2$, where T_1 is the time taken to empty the tank when initially it is completely full, and T_2 is the time taken to empty the tank when initially it is half full. [4]

8 A curve C has equation $y^2 + xy = 4$, where $y > 0$.

(i) Without using a calculator, find the coordinates of the point on C at which the gradient is $-\frac{1}{5}$. [4]

(ii) Variables z and y are related by the equation $y^2 + z^2 = 10y$, where $z > 0$. Given that x increases at a constant rate of 0.5 unit/s, find the rate of change of z when $x = 3$. [5]

9 **(a)** The complex numbers z and w satisfy the simultaneous equations

$$|z| - w^* = -3 - \sqrt{2}\text{i} \text{ and } w^* + w + 5z = 1 + 20\text{i},$$

where w^* is the complex conjugate of w. Find the value of z and the corresponding value of w. [4]

(b) It is given that 8i is a root of the equation $\text{i}z^3 + (8 - 2\text{i})z^2 + az + 40 = 0$ where a is a complex number.

(i) Find a. [2]

(ii) Hence, find the other roots of the equation, leaving your answer in the form $a + b\text{i}$ where a and b are real constants. [3]

(iii) Deduce the number of real roots the equation $z^3 - (8 - 2\text{i})z^2 + a\text{i}z + 40 = 0$ has. [1]

10 For this question, you may use the results $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

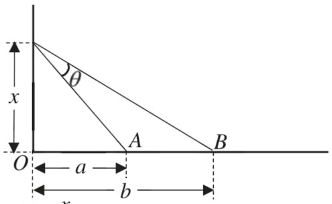
(i) Find $\displaystyle\sum_{r=1}^{n} r^2(2r-1)$ in terms of n. [2]

(ii) Find $\displaystyle\sum_{r=1}^{n} r^2(r-1)$ in terms of n. Hence find $\displaystyle\sum_{r=2}^{n-1} r(r+1)^2$ in terms of n. [5]

(iii) Without using a graphing calculator, find the sum of the series

$$4(25) - 5(36) + 6(49) - 7(64) + \cdots - 59(3600).$$ [3]

11 A boy is playing a ball game on a field. He arranges two cones A and B along the end of the field such that the cones are a and b metres respectively from one corner, O, of the field as shown in the diagram below. The boy stands along the edge of the field at x metres from O and kicks the ball between the two cones. The angle that the two cones subtends at the position of the boy is denoted by θ.



(i) Show that $\tan\theta = (b-a)\dfrac{x}{x^2+ab}$. [2]

(ii) It is given that $a = 15$ and $b = 20$. Find by differentiation, the value of x such that θ is at a maximum. [3]

(iii) It is given instead that the boy gets two friends to vary the position of both cones A and B along the end of the field such that $5 \le a \le 12$ and $b = 2a$, and the boy moves along the edge of the field such that his distance from cone A remains unchanged at 18 metres. Sketch a graph that shows how θ varies with a and find the largest possible value of θ. [4]

(iv) The boy runs until he is at a distance k metres from the goal line that is formed by the two cones and kicks the ball toward the goal line. The path of the ball is modelled by the equation $h = -\left(\dfrac{1}{10}k+2\right)^2 + 6$, where k is the distance of the ball from the goal line and h its corresponding height above the ground respectively. Find the angle that the path of the ball makes with the horizontal at the instant the ball crosses the goal line. [3]

12 In the study of force field, we are often interested in whether the work done in moving an object from point A to point B is independent of the path taken. If a force field is such that the work done is independent of the path taken, it is said to be a *conservative* field.

A force field **F** can be regarded as a vector $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ where M and N are functions of x and y. The path that the object is moving along is denoted by C. The work done in moving the object along the curve C from the point where $x = a$ to the point where $x = b$ is given by

$$W = \int_a^b \left[M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} \right] \mathrm{d}x,$$

where $y = \mathrm{f}(x)$ is the equation of the curve C.

(i) Sketch the curve C with equation $y^2 = 4(1-x)$, for $x \le 1$. [2]

(ii) Find an expression of $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y. [1]

(iii) The points P and Q are on C with $x = 1$ and $x = -3$ respectively and Q is below the x-axis. Find the equation of the line PQ. [2]

For the rest of the question, the force field is given by $\mathbf{F} = x^2\mathbf{i} + xy^2\mathbf{j}$.

(iv) Show that the work done in moving an object along the curve C from Q to P is given by the integral $\int_{-3}^{1} \left(x^2 + 4x(1-x)^{\frac{1}{2}} \right) \mathrm{d}x$. Hence evaluate the exact work done in moving the object along the curve C from Q to P. [4]

(v) Find the work done in moving an object along the line PQ from Q to P to 2 decimal places. [2]

(vi) Determine, with reason, whether **F** is a conservative force field. [1]

–––––END OF PAPER–––––

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

Candidate Name					
CT Class	1	8			

Centre Number/ Index Number: ____ ____ ____ ____ / ____ ____ ____ ____

MATHEMATICS 9758/02

Paper 2

16th September 2019

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
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You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	**Mark**
1	
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Total	

Section A: Pure Mathematics [40 marks]

1 **(i)** Show that $\frac{1}{(n-1)!}-\frac{3}{n!}+\frac{2}{(n+1)!}=\frac{An^2+Bn+C}{(n+1)!}$ where A, B and C are constants to be determined. [2]

(ii) Hence find $\sum_{n=1}^{N}\frac{n^2-2n-1}{5(n+1)!}$ in terms of N. [3]

(iii) Give a reason why the series $\sum_{n=1}^{\infty}\frac{n^2-2n-1}{5(n+1)!}$ converges and write down its value. [2]

2 The curve C has parametric equations $x=6t^2$, $y=\frac{2t}{\sqrt{1-t^2}}$, $0<t<1$.

(i) A line is tangent to the curve C at point A and passes through the origin O. Show that the line has equation $y=\frac{2}{3}x$. [4]

The region R is bounded by the curve and the tangent line in **(i)**.

(ii) Find the area of R. [3]

(iii) Write down the Cartesian equation of the curve C. [1]

(iv) Find the exact volume of the solid of revolution generated when R is rotated completely about the x-axis, giving your answer in the form $(a\ln b-c)\pi$, where constants a, b, c are to be determined. [4]

3 When a ball is dropped from a height of H m above the ground, it will rebound to a height of eH m where $0<e<1$. The height of each successive bounce will be e times of that of its previous height. It is also known that the time taken between successive bounce is given by $t=0.90305\sqrt{h}$ where h is the maximum height of the ball from the ground between these bounces. We can assume that there is negligible air resistance.
A ball is now dropped from a height of 10 m from the ground. Let t_n be the time between the n^{th} and $(n+1)^{\text{th}}$ bounce.

(i) Show that the total distance travelled by the ball just before the n^{th} bounce is $\frac{10(1+e-2e^n)}{1-e}$. [3]

(ii) Show that t_n is a geometric sequence. State the common ratio for this sequence. [3]

(iii) Find in terms of e the total distance the ball will travel and the time taken when it comes to rest. You may assume that between any two bounces, the time taken for the ball to reach its maximum height is the same as the time it takes to return to the ground. [3]

4 Referred to the origin, the points A and B have position vectors $-\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $3\mathbf{i}+\mathbf{k}$ respectively. The plane π has equation $\mathbf{r}=\begin{pmatrix}5\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\2\\a\end{pmatrix}$, and the line l has equation $\mathbf{r}=\begin{pmatrix}3\\0\\a\end{pmatrix}+t\begin{pmatrix}4a\\4\\1\end{pmatrix}$, where a is a constant and λ, μ, and t are parameters.

(i) Show that for all real values of a, l is parallel to π. [2]

(ii) Find the value of a such that l and π have common points. [2]

For the rest of the question, let $a=1$.

(iii) Find the projection of $\overrightarrow{AB}$ onto π. [3]

(iv) Let F be the foot of perpendicular from A to π. The point C lies on AF extended such that $\angle ABF=\angle CBF$. Find a cartesian equation of the plane that contains C and l. [3]

(v) Let D be a point on l. Find the largest possible value of the non-reflex angle $\angle ADC$. [2]

Section B: Probability and Statistics [60 marks]

5 This question is about arrangements of all nine letters in the word ADDRESSEE.

(i) Find the number of different arrangements of the nine letters. [1]

(ii) Find the number of different arrangements that can be made with both the D's together and both the S's together. [2]

(iii) Find the number of different arrangements that can be made where the E's are separated by at least one letter and the D's are together. [2]

(iv) Find the number of different arrangements that can be made where the E's are not together, S's are not together and the D's are not together. [4]

6 Emergency flares are simple signalling devices similar to fireworks and they are designed to communicate a much more direct message in an emergency, for example, distress at sea.

A company categorised their stocks of emergency flares as 1-year old, 5-year old and 10-year old. The probabilities of successful firing of 1-year old, 5-year old and 10-year old emergency flares are 0.995, 0.970 and 0.750 respectively.

(i) Find the probability that, out of 100 randomly chosen 1-year old flares, at most 2 fail to fire successfully. [1]

(ii) One-year old flares are packed into boxes of 100 flares. Find the probability that, out of 50 randomly chosen boxes of 1-year old flares, not more than 48 of these boxes will have at most 2 flares that will fail to fire successfully in each box. [3]

(iii) Seven flares are chosen at random, of which one is 5 years old and six are 10 years old. Find the probability that

(a) the 5-year old flare fails to fire successfully and at least 4 of the 10-year old flares fire successfully, [2]

(b) at least 4 of the 7 flares fire successfully. [3]

7 With the move towards automated services at a bank, only two cashiers will be deployed to serve customers wanting to withdraw or deposit cash. For each cashier, the bank observed that the time taken to serve a customer is a random variable having a normal distribution with mean 150 seconds and standard deviation 45 seconds.

(i) Find the probability that the time taken for a randomly chosen customer to be served by a cashier is more than 180 seconds. [1]

(ii) One of the two cashiers serves two customers, one straight after the other. By stating a necessary assumption, find the probability that the total time taken by the cashier is less than 200 seconds. [3]

(iii) During peak-hour on a particular day, one cashier has a queue of 4 customers and the other cashier has a queue of 3 customers, and the cashiers begin to deal with customers at the front of their queues. Assuming that the time taken by each cashier to serve a customer is independent of the other cashier, find the probability that the 4 customers in the first queue will all be served before the 3 customers in the second queue are all served. [3]

8 To study if the urea serum content, u mmol per litre, depends on the age of a person, 10 patients of different ages, x years, admitted into the Accident and Emergency Department of a hospital are taken for study by a medical student. The results are shown in the table below.

Age, x (years)	37	44	56	60	64	71	74	77	81	89
Urea, u (mmol/l)	4.2	5.1	4.9	5.7	7.4	7.0	6.8	6.2	7.8	9.6

(i) Draw a scatter diagram of these data. [1]

(ii) By calculating the relevant product moment correlation coefficients, determine whether the relationship between u and x is modelled better by $u = ax + b$ or by $u = ae^{bx}$. Explain how you decide which model is better, and state the equation in this case. [5]

(iii) Explain why we can use the equation in **(ii)** to estimate the age of the patient when the urea serum is 7 mmol per litre. Find the estimated age of the patient when the urea serum is 7 mmol per litre [2]

(iv) The units for the urea serum is now given in mmol per decilitre.

(a) Give a reason if the product moment correlation coefficient calculated in **(ii)** will be changed. [1]

(b) Given that 1 decilitre is equal to 0.1 litre, re-write your equation in **(ii)** so that it can be used when the urea serum is given in mmol per decilitre. [1]

9 A game is played with 18 cards, each printed with a number from 1 to 6 and each number appears on exactly 3 cards. A player draws 3 cards without replacement. The random variable X is the number of cards with the same number.

(i) Show that $\mathrm{P}(X=2)=\dfrac{45}{136}$ and determine the probability distribution of X. [3]

(ii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X)=0.922$ correct to 3 significant figures. [3]

(iii) 40 games are played. Find the probability that the average number of cards with the same number is more than 1. [2]

(iv) In each game, Sam wins $\$(a+10)$ if there are cards with the same number, otherwise he loses $\$a$. Find the possible values of a, where a is an integer, such that Sam's expected winnings per game is positive. [4]

10 In the manufacturing of a computer device, there is a process which coats a computer part with a material that is supposed to be 100 microns thick. If the coating is too thin, the proper insulation of the computer device will not occur and it will not function reliably. Similarly, if the coating is too thick, the device will not fit properly with other computer components.

The manufacturer has calibrated the machine that applies the coating so that it has an average coating depth of 100 microns with a standard deviation of 10 microns. When calibrated this way, the process is said to be "in control".

Due to wear out of mechanical parts, there is a tendency for the process to drift. Hence the process has to be monitored to make sure that it is in control.

(i) After running the process for a reasonable time, a random sample of 50 computer devices is drawn. The sample mean is found to be 103.4 microns. Test at the 5% level of significance whether the sample suggests that the process is not in control. State any assumptions for this test to be valid. [4]

(ii) To ease the procedure of checking, the supervisor of this process would like to find the range of values of the sample mean of a random sample of size 50 that will suggests that the process is not in control at 5% level of significance. Find the required range of values of the sample mean, leaving your answer to 1 decimal places. [3]

On another occasion, a random sample of 40 computer devices is taken. The data can be summarised by

$$\Sigma(y-100)=164, \quad \Sigma(y-100)^2=9447.$$

(iii) Calculate the unbiased estimate for the population mean and population variance of the thickness of a coating on the computer device. [2]

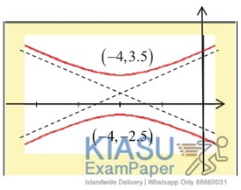
(iv) Give, in context, a reason why we may not be able to use 10 microns for the standard deviation of the thickness of a coating on the computer device. [1]

(v) Assume that the standard deviation has changed, test at the 4% level of significance whether the sample suggests that the process is not in control. [3]

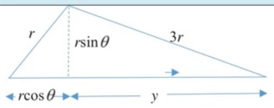
–––––END OF PAPER–––––

Dr.Kenny Education

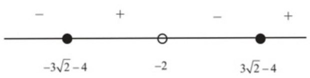
<u>**2019 NYJC JC2 Prelim 9758/1 Solution**</u>

Qn	
1	$y^2 = x^2 + 9$ Scale 2 parallel to x $y^2 = \left(\frac{x}{2}\right)^2 + 9$ Translate by -4 units parallel to x $y^2 = \left(\frac{(x+4)}{2}\right)^2 + 9$ Translate by 1/2 units parallel to y $\left(y - \frac{1}{2}\right)^2 = \left(\frac{(x+4)}{2}\right)^2 + 9$ Equations of asymptotes $y = \frac{1}{2} \pm \frac{x+4}{2} = \frac{x}{2} + \frac{5}{2}, -\frac{x}{2} - \frac{3}{2}$  $y = \frac{x}{2} + \frac{5}{2}$ $y = -\frac{x}{2} - \frac{3}{2}$

2019 NYJC JC2 Prelim 9758/1 Solution

<table>
<tr><th>Qn</th><th></th></tr>
<tr><td>2(i)</td><td>
Using Pythagoras' Theorem, $y=\sqrt{9r^2-r^2\sin^2\theta}$
$\therefore x=r\cos\theta+r\sqrt{9-\sin^2\theta}\;=x=r\left[\cos\theta+\sqrt{\left(9-\sin^2\theta\right)}\right]$</td></tr>
<tr><td>2(ii)</td><td>Max $x=4r$</td></tr>
<tr><td>2(iii)</td><td>$x\approx r\left[\left(1-\frac{\theta^2}{2}\right)+\left(9-\theta^2\right)^{\frac{1}{2}}\right]$
$\approx r\left[1-\frac{\theta^2}{2}+3\left(1-\frac{\theta^2}{9}\right)^{\frac{1}{2}}\right]$
$\approx r\left[1-\frac{\theta^2}{2}+3\left(1-\frac{1}{2}\left(\frac{\theta^2}{9}\right)\right)\right]$
$=r\left(4-\frac{2}{3}\theta^2\right)$
</td></tr>
</table>

2019 NYJC JC2 Prelim 9758/1 Solution

Qn	
3	$\frac{x^2-3x+4}{x+2} \geq 2x+1$ $\frac{x^2-3x+4-(2x+1)(x+2)}{x+2} \geq 0$ $\frac{-x^2-8x+2}{x+2} \geq 0$ $\frac{x^2+8x-2}{x+2} \leq 0$ $\frac{(x+4)^2-18}{x+2} \leq 0$ $\frac{(x+4-3\sqrt{2})(x+4+3\sqrt{2})}{x+2} \leq 0$  $x \leq -3\sqrt{2}-4$ or $-2 < x \leq 3\sqrt{2}-4$ Replacing x by $-a^x$ $\frac{a^{2x}+3a^x+4}{-a^x+2} \geq -2a^x+1$ $\frac{a^{2x}+3a^x+4}{a^x-2} \leq 2a^x-1$ Since $-a^x < 0$ $-a^x \leq -3\sqrt{2}-4$ or $-2 < -a^x \leq 3\sqrt{2}-4$ $x \geq \frac{\ln(3\sqrt{2}+4)}{\ln a}$ or $x < \frac{\ln 2}{\ln a}$

Dr.Kenny Education

Qn	
4(i)	$\sin^4\theta = \frac{1}{4}\left(2\sin^2\theta\right)^2$ $= \frac{1}{4}\left(1-\cos 2\theta\right)^2$ $= \frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)$ $= \frac{1}{4}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)$ $= \frac{1}{8}\left(3-4\cos 2\theta+\cos 4\theta\right)$
4(ii)	Let $x = 2\cos\theta$. Thus $\frac{dx}{d\theta} = -2\sin\theta$. When $x=0$, $\theta = \frac{\pi}{2}$; when $x=2$, $\theta=0$. $\int_0^2 \left(4-x^2\right)^{\frac{3}{2}}\,dx = \int_{\frac{\pi}{2}}^0 \left(4-4\cos^2\theta\right)^{\frac{3}{2}}(-2\sin\theta)\,d\theta$ $= 2\int_0^{\frac{\pi}{2}} \sin\theta\left(4\sin^2\theta\right)^{\frac{3}{2}}\,d\theta$ $= 16\int_0^{\frac{\pi}{2}} \sin^4\theta\,d\theta$ $= 2\int_0^{\frac{\pi}{2}} \left(3-4\cos 2\theta+\cos 4\theta\right)d\theta$ $= 2\left[3\theta-2\sin 2\theta+\frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$ $= 3\pi$

2019 NYJC JC2 Prelim 9758/1 Solution

Qn	
5(i)	$\overrightarrow{OD} = \lambda b + (1-\lambda)a, \quad \lambda \in \mathbb{R}$ $l_{OD} : r = s(\lambda b + (1-\lambda)a), \qquad s \in \mathbb{R}$
5(ii)	$\overrightarrow{OE} = s(\lambda \mathbf{b} + (1-\lambda)\mathbf{a}), \qquad$ for some $s, \lambda \in \mathbb{R}$. $\overrightarrow{OE} = \frac{1}{2}(\mathbf{b} + 3\mathbf{a})$ $s(\lambda \mathbf{b} + (1-\lambda)\mathbf{a}) = \frac{1}{2}(\mathbf{b} + 3\mathbf{a})$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel ($\lambda > 0$), $s\lambda = \frac{1}{2}$ $s(1-\lambda) = \frac{3}{2}$ Solving, $\lambda = \frac{1}{4}$. **Area of** $\Delta BED = \frac{1}{2}\left\|\overrightarrow{BE} \times \overrightarrow{BD}\right\|$ $= \frac{1}{2}\|(-\frac{1}{2}\mathbf{b} + \frac{3}{2}\mathbf{a}) \times (-\frac{3}{4}\mathbf{b} + \frac{3}{4}\mathbf{a})\|$ $= \frac{1}{2}\|\frac{3}{8}\mathbf{b} \times \mathbf{b} - \frac{3}{8}\mathbf{b} \times \mathbf{a} - \frac{9}{8}\mathbf{a} \times \mathbf{b} + \frac{9}{8}\mathbf{a} \times \mathbf{a}\|$ $= \frac{1}{2}\left\|\frac{3}{8}\mathbf{a} \times \mathbf{b} - \frac{9}{8}\mathbf{a} \times \mathbf{b}\right\|$ $= \frac{3}{8}\|\mathbf{a} \times \mathbf{b}\|$ $k = \frac{3}{8}$

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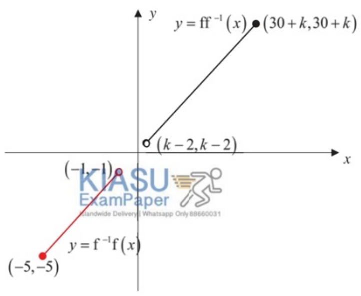
Qn	
6(i)	$\mathrm{f}(x)=2x^2+4x+k=2(x+1)^2+k-2$ For f^{-1} to exist, f must be one-one. Largest value of $a=-1$
6(ii)	Let $y=\mathrm{f}(x)$ $y=2(x+1)^2+k-2$ $x=-1\pm\sqrt{\frac{1}{2}(y-k+2)}$ Since $x\in[-5,-1)$, $x<-1$ Hence $x=-1-\sqrt{\frac{1}{2}(y-k+2)}$ For $-5\le x<a$, $k-2<\mathrm{f}(x)\le 30+k$, $\mathrm{f}(x)=-1-\sqrt{\frac{1}{2}(x-k+2)}$, $D_{\mathrm{f}^{-1}}=(k-2,30+k]$
6(iii)	 Number of solutions to $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$ is 0.

<table>
<tr><th>Qn</th><th></th></tr>
<tr><td>7(i)</td><td>$V=\frac{1}{3}\pi x^2\left(3a-x\right)\Rightarrow\frac{\mathrm{d}V}{\mathrm{d}t}=\left(2\pi ax-\pi x^2\right)\frac{\mathrm{d}x}{\mathrm{d}t}$
Since $\frac{\mathrm{d}V}{\mathrm{d}t}=-\pi k\sqrt{x}$,
$\left(2\pi ax-\pi x^2\right)\frac{\mathrm{d}x}{\mathrm{d}t}=-\pi k\sqrt{x}$
$\left(2ax-x^2\right)\frac{\mathrm{d}x}{\mathrm{d}t}=-k\sqrt{x}$</td></tr>
<tr><td>7(ii)</td><td>$\int\frac{2ax-x^2}{\sqrt{x}}\,\mathrm{d}x=\int -k\,\mathrm{d}t$
$\Rightarrow\int 2a\sqrt{x}-x^{\frac{3}{2}}\,\mathrm{d}x=-kt+c$
$\Rightarrow\frac{4}{3}ax^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}=-kt+c$
$\Rightarrow t=\frac{1}{k}\left[c-\frac{4}{3}ax^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}\right]$</td></tr>
<tr><td>7(iii)</td><td>If the tank is initially full, $x=2a$, thus
$c=\frac{4}{3}a(2a)^{\frac{3}{2}}-\frac{2}{5}(2a)^{\frac{5}{2}}=\frac{16}{15}a^2\sqrt{2a}$
Thus $T_1=\frac{c}{k}=\frac{16}{5k}a^2\sqrt{2a}$
If the tank is initially half full, $x=a$, thus
$c=\frac{4}{3}a(a)^{\frac{3}{2}}-\frac{2}{5}(a)^{\frac{5}{2}}=\frac{14}{15}a^2\sqrt{a}$
Thus $T_2=\frac{c}{k}=\frac{14}{5k}a^2\sqrt{a}$
Thus $\frac{T_1}{T_2}=\frac{16a^2\sqrt{2a}}{14a^2\sqrt{a}}=\frac{8\sqrt{2}}{7}$
Required ratio is $8\sqrt{2}:7$</td></tr>
</table>

Qn	
8(i)	$y^2 + xy = 4$ ___(1) Differentiate w.r.t. x, $2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$ $(2y + x)\frac{dy}{dx} = -y$ ___(2) $\frac{dy}{dx} = -\frac{y}{2y+x}$ When $\frac{dy}{dx} = -\frac{y}{2y+x} = -\frac{1}{5}$ $5y = 2y + x$ $x = 3y$ Substitute $x = 3y$ in (1), $y^2 + 3y^2 = 4$ $y^2 = 1$ Hence $y = 1$ ($\because y > 0$) Coordinates of the point are $(3,1)$
8(ii)	$y^2 + z^2 = 10y$ ___(3) Differentiate (3) with respect to y, $2y + 2z\frac{dz}{dy} = 10$ $y + z\frac{dz}{dy} = 5$ $\frac{dz}{dy} = \frac{5-y}{z}$ $\frac{dz}{dt} = \frac{dz}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dt}$ $= \frac{5-y}{z}\left(-\frac{y}{2y+x}\right)\frac{dx}{dt}$

Qn	

At $x=3$, $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{5}$ from (i).

From (3), $1^2+z^2=10$

$z=3\ (\because z>0)$

Hence $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{5-1}{3}\left(-\frac{1}{5}\right)\frac{1}{2}=-\frac{2}{15}$

<u>Alternatively,</u>

$y^2+z^2=10y$ ___(3)

Differentiate (3) with respect to y,

$$2y+2z\frac{\mathrm{d}z}{\mathrm{d}y}=10$$

$$y+z\frac{\mathrm{d}z}{\mathrm{d}y}=5$$

$$\frac{\mathrm{d}z}{\mathrm{d}y}=\frac{5-y}{z}$$

From (2), $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}$

$$=-\frac{y}{2y+x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

At $x=3$, $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{5}$ from (i).

$$\frac{\mathrm{d}y}{\mathrm{d}t}=-\frac{1}{5}\cdot\frac{1}{2}=-\frac{1}{10}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\mathrm{d}z}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$=\frac{5-y}{z}\frac{\mathrm{d}y}{\mathrm{d}t}$$

From (3), $1^2+z^2=10$

$z=3\ (\because z>0)$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{5-1}{3}\left(-\frac{1}{10}\right)=-\frac{2}{15}$$

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Qn	
9(a)	$\lvert z\rvert - w^* = -3-\sqrt{2}\mathrm{i}$ $\Rightarrow w^* = \lvert z\rvert + 3 + \sqrt{2}\mathrm{i}$ and $w = \lvert z\rvert + 3 - \sqrt{2}\mathrm{i}$ Sub into $w^* + w + 5z = 1 + 20\mathrm{i}$, Let $z = x + y\mathrm{i}$, where x and y are real. $2\sqrt{x^2+y^2} + 5x + 5y\mathrm{i} = -5 + 20\mathrm{i}$ Comparing real and imaginary components, $2\sqrt{x^2+y^2} + 5x = -5$, $5y = 20 \Rightarrow y = 4$ $2\sqrt{x^2+16} + 5x = -5$ $2\sqrt{x^2+16} = -5x - 5$ $4\left(x^2+16\right) = 25x^2 + 50x + 25$ $21x^2 + 50x - 39 = 0$ $x = \frac{13}{21}$ or $x = -3$ (reject $x = \frac{13}{21} \because 2\sqrt{x^2+y^2} + 5x = -5$) $z = -3 + 4\mathrm{i}$, $w = 8 - \sqrt{2}\mathrm{i}$
9(b) **(i)**	$\mathrm{i}(8\mathrm{i})^3 + (8-2\mathrm{i})(8\mathrm{i})^2 + a(8\mathrm{i}) + 40 = 0$ $512 - 64(8-2\mathrm{i}) + 8a\mathrm{i} + 40 = 0$ $a\mathrm{i} = -5 - 16\mathrm{i} \Rightarrow a = -16 + 5\mathrm{i}$
9(b) **(ii)**	$(z-8\mathrm{i})\left(Az^2 + Bz + C\right) = 0$ Comparing coefficient for z^3, $A = \mathrm{i}$ Comparing coefficient for constant, $C = 5\mathrm{i}$ Comparing coefficient for z^2, $B - 8\mathrm{i}A = 8 - 2\mathrm{i}$ $B = -2\mathrm{i}$

Qn	
	$(z-8i)(iz^2-2iz+5i)=0$ $(z-8i)(z^2-2z+5)=0$ The other roots are $z=\frac{2\pm\sqrt{4-20}}{2}$ $=1\pm 2i$
9(b) (iii)	Replacing z with iz, $iz=8i$ or $iz=1\pm 2i$ $z=8$ $\quad z=\pm 2-i$ Therefore 1 real root.

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Qn	
10(i)	$\sum_{r=1}^{n} r^2(2r-1) = \sum_{r=1}^{n}(2r^3 - r^2)$ $= \frac{2}{4}n^2(n+1)^2 - \frac{n}{6}(n+1)(2n+1)$ $= \frac{1}{6}n(n+1)\left[3n(n+1)-(2n+1)\right]$ $= \frac{1}{6}n(n+1)(3n^2+n-1)$
10(ii)	$\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n}(r^3 - r^2)$ $= \frac{1}{4}n^2(n+1)^2 - \frac{n}{6}(n+1)(2n+1)$ $= \frac{1}{12}n(n+1)\left[3n(n+1)-2(2n+1)\right]$ $= \frac{1}{12}n(n+1)(3n^2-n-2)$ $= \frac{1}{12}n(n+1)(3n+2)(n-1)$ $\sum_{r=2}^{n-1} r(r+1)^2 = \sum_{k-1=2}^{k-1=n-1}(k-1)k^2$ $= \sum_{k=3}^{k=n} k^2(k-1)$ $= \sum_{k=1}^{k=n} k^2(k-1) - \sum_{k=1}^{k=2} k^2(k-1)$ $= \frac{1}{12}n(n+1)(n-1)(3n+2) - \frac{1}{12}(2)(3)(1)(8)$ $= \frac{1}{12}n(n+1)(n-1)(3n+2) - 4$

Qn	
10(iii)	**(iii)** $4(25)-5(36)-\cdots-59(3600)$ $=4(25)+5(36)+\cdots+59(3600)-2\left[5(36)+7(64)\cdots+59(3600)\right]$ $=\sum_{r=5}^{60} r^2(r-1)-2\sum_{r=3}^{30}(2r)^2(2r-1)$ $=\sum_{r=1}^{60} r^2(r-1)-\sum_{r=1}^{4} r^2(r-1)-2\sum_{r=1}^{30}(2r)^2(2r-1)+2\sum_{r=1}^{2}(2r)^2(2r-1)$ $=\frac{1}{12}(60)(61)(59)(182)-\frac{1}{12}(4)(5)(3)(14)$ $-\frac{4}{3}(30)(31)(2729)+\frac{4}{3}(2)(3)(13)$ $=-108836$

Dr.Kenny Education

Qn	
11(i)	Denote the position of the boy by X.

Let $\angle OXA = \alpha$ and $\angle OXB = \beta$. Then $\theta = \beta - \alpha$ and

$$\tan\theta = \tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

$$= \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x}\cdot\frac{a}{x}}$$

$$= \frac{\left(\frac{b-a}{x}\right)x^2}{\left(1 + \frac{ab}{x^2}\right)x^2} = (b-a)\frac{x}{x^2 + ab}$$

Alternatively:

Applying sine rule,

$$\frac{\sin\theta}{b-a} = \frac{\sin B}{\sqrt{x^2 + a^2}} \quad \Rightarrow \quad \sin\theta = \frac{(b-a)}{\sqrt{x^2 + a^2}}\sin B$$

$$= \frac{(b-a)}{\sqrt{x^2 + a^2}}\cdot\frac{x}{\sqrt{x^2 + b^2}}$$

Applying cosine rule,

$$(b-a)^2 = (x^2 + a^2) + (x^2 + b^2) - 2\sqrt{x^2 + a^2}\sqrt{x^2 + b^2}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{(x^2 + a^2) + (x^2 + b^2) - (b-a)^2}{2\sqrt{x^2 + a^2}\sqrt{x^2 + b^2}}$$

$$= \frac{x^2 + ab}{\sqrt{x^2 + a^2}\sqrt{x^2 + b^2}}$$

Hence $\tan\theta = \frac{(b-a)}{\sqrt{x^2 + a^2}}\cdot\frac{x}{\sqrt{x^2 + b^2}} \bigg/ \frac{x^2 + ab}{\sqrt{x^2 + a^2}\sqrt{x^2 + b^2}}$

$$= (b-a)\frac{x}{x^2 + ab}$$

Qn	
(ii)	Differentiate $\tan\theta = \dfrac{5x}{x^2+300}$ with respect to x: $\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{5\left(x^2+300\right) - 5x \cdot 2x}{\left(x^2+300\right)^2}$ $= \dfrac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0 \Rightarrow x^2 = 300$ $x = \sqrt{300}$ or 17.3 (3s.f.) <u>Alternatively</u>, Differentiate $\tan\theta\left(x^2+300\right) = 5x$ with respect to x: $\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}x}\left(x^2+300\right) + \tan\theta(2x) = 5$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0 \Rightarrow \tan\theta(2x) = 5$ Substitute $\tan\theta = \dfrac{5}{2x}$ into $\tan\theta = \dfrac{5x}{x^2+300}$: $\dfrac{5}{2x} = \dfrac{5x}{x^2+300}$ $x^2 + 300 = 2x^2$ $x^2 = 300$ $x = \sqrt{300}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2 \sec^2\theta}$.

Qn	
	Since $\left(x^2+300\right)^2 \sec^2\theta > 0$ for any x and θ, it suffices to test the sign of $-x^2+300$.

x	$\left(\sqrt{300}\right)^-$	$\sqrt{300}$	$\left(\sqrt{300}\right)^+$
$\dfrac{\mathrm{d}\theta}{\mathrm{d}x}=\dfrac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\sec^2\theta}$	>0	=0	<0

Hence θ is maximum

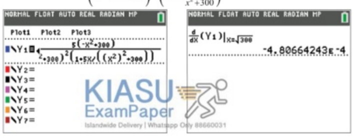
<u>Alternatively</u>, apply second derivative test:

Using GC:

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\sec^2\theta}$$

$$=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\left(1+\tan^2\theta\right)}$$

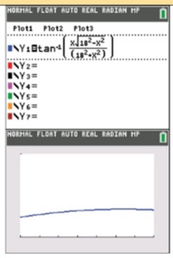
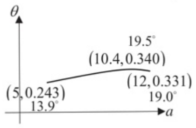
$$=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2\left(1+\frac{5x}{x^2+300}\right)}$$

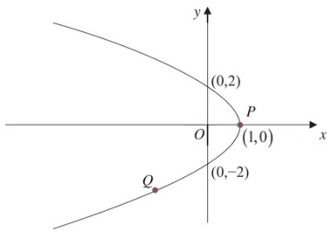


$\left.\dfrac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\right|_{x=\sqrt{300}}=-4.81\times10^{-4}<0$. Hence θ is maximum.

Or:

Qn	
	Differentiate $\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{5\left(-x^2+300\right)}{\left(x^2+300\right)^2}$ with respect to x: $2\sec\theta\left(\sec\theta\tan\theta\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}x}+\sec^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}$ $=\frac{5(-2x)\left(x^2+300\right)^2-5\left(-x^2+300\right)\cdot 2\cdot 2x\left(x^2+300\right)}{\left(x^2+300\right)^4}$ $=\frac{-10x\left(x^2+300\right)\left[\left(x^2+300\right)+2\left(-x^2+300\right)\right]}{\left(x^2+300\right)^4}$ $=\frac{-10x\left(x^2+300\right)\left[-x^2+900\right]}{\left(x^2+300\right)^4}$ At $x=\sqrt{300}$, $\frac{\mathrm{d}\theta}{\mathrm{d}x}=0$, $-x^2+900=-300+900=600$ $\frac{-10x\left(x^2+300\right)(600)}{\left(x^2+300\right)^4}<0$, and $\sec^2\theta>0$, Hence $\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}<0$ and θ is maximum.
(iii)	Since $b=2a$, $\tan\theta=(2a-a)\frac{x}{x^2+a(2a)}$ $\tan\theta=\frac{ax}{x^2+2a^2}$ $x^2+a^2=18^2$ $x^2=18^2-a^2$ Hence $\tan\theta=\frac{a\sqrt{18^2-a^2}}{18^2-a^2+2a^2}$ $\theta=\tan^{-1}\left(\frac{a\sqrt{18^2-a^2}}{18^2+a^2}\right)$

Qn	
	   Largest possible $\theta = 0.340$ rad (3 s.f.) or $\theta = 19.5^\circ$ (1 d.p.)
(iv)	Differentiate $h = -\left(\frac{1}{10}k + 2\right)^2 + 6$ with respect to k: $\frac{\mathrm{d}h}{\mathrm{d}k} = -\frac{2}{10}\left(\frac{1}{10}k + 2\right)$ At the instant when the ball crosses the goal line, $k = 0$ $\frac{\mathrm{d}h}{\mathrm{d}k} = -\frac{2}{5}$ $\tan\phi = -\frac{2}{5}$ $\phi = -0.381$ (3 s.f.) or $\theta \approx -21.8^\circ$ (1 d.p.)

Qn	
12(i)	
12(ii)	$y^2 = 4(1-x)$ Differentiate wrt x: $2y\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{y}$
12(iii)	At P, $x=1$, $y=0$; at Q, $x=-3$, $y=-4$. Thus equation of line PQ is $\frac{y}{x-1} = \frac{0-(-4)}{1-(-3)} \quad \Rightarrow \quad y = x-1$

<table>
<tr><th>Qn</th><th></th></tr>
<tr><td>12(iv)</td><td>
Along arc QP, $y=-2\sqrt{1-x}$.

$$W_C=\int_{-3}^{1}\left(x^2+xy^2\cdot\left(\frac{-2}{y}\right)\right)\mathrm{d}x$$
$$=\int_{-3}^{1}\left(x^2-2xy\right)\mathrm{d}x$$
$$=\int_{-3}^{1}\left(x^2+4x(1-x)^{\frac{1}{2}}\right)\mathrm{d}x$$

<table>
<tr><td>Method 1

$$W_C=\left[\frac{x^3}{3}\right]_{-3}^{1}-\left[\frac{8}{3}x(1-x)^{\frac{3}{2}}\right]_{-3}^{1}$$
$$+\int_{-3}^{1}\frac{8}{3}(1-x)^{\frac{3}{2}}\,\mathrm{d}x$$
$$=\frac{28}{3}-64-\left[\frac{16}{15}(1-x)^{\frac{5}{2}}\right]_{-3}^{1}$$
$$=-\frac{164}{3}+\frac{512}{15}=-\frac{308}{15}$$
</td><td>Method 2

$$W_C=\int_{-3}^{1}\left(x^2+4x(1-x)^{\frac{1}{2}}\right)\mathrm{d}x$$
$$=\int_{-3}^{1}\left(x^2-4(1-x)(1-x)^{\frac{1}{2}}+4(1-x)^{\frac{1}{2}}\right)\mathrm{d}x$$
$$=\left[\frac{x^3}{3}+\frac{8}{5}(1-x)^{\frac{5}{2}}-\frac{8}{3}(1-x)^{\frac{3}{2}}\right]_{-3}^{1}$$
$$=\frac{28}{3}-\frac{256}{5}+\frac{64}{3}=-\frac{308}{15}$$
</td></tr>
</table>
</td></tr>
<tr><td>12(v)</td><td>

$$W_L=\int_{-3}^{1}\left(x^2+xy^2\right)\mathrm{d}x=\int_{-3}^{1}\left(x^2+x(x-1)^2\right)\mathrm{d}x$$
$$=-33.33$$
</td></tr>
<tr><td>12(vi)</td><td>Since the work done for the two paths are different, the force field F is not conservative.</td></tr>
</table>

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Qn	
1(i)	$\frac{1}{(n-1)!}-\frac{3}{n!}+\frac{2}{(n+1)!}=\frac{n(n+1)-3(n+1)+2}{(n+1)!}$ $=\frac{n^2+n-3n-3+2}{(n+1)!}=\frac{n^2-2n-1}{(n+1)!}$ Hence $A=1, B=-2, C=-1$
1(ii)	$\sum_{n=1}^{N}\frac{n^2-2n-1}{5(n+1)!}=\frac{1}{5}\sum_{n=1}^{N}\frac{n^2-2n-1}{(n+1)!}=\frac{1}{5}\sum_{n=1}^{N}\left[\frac{1}{(n-1)!}-\frac{3}{n!}+\frac{2}{(n+1)!}\right]$ $=\frac{1}{5}\left[\begin{array}{c}\frac{1}{0!}-\frac{3}{1!}+\frac{2}{2!}\\+\frac{1}{1!}-\frac{3}{2!}+\frac{2}{3!}\\+\frac{1}{2!}-\frac{3}{3!}+\frac{2}{4!}\\+\vdots\\+\frac{1}{(N-3)!}-\frac{3}{(N-2)!}+\frac{2}{(N-1)!}\\+\frac{1}{(N-2)!}-\frac{3}{(N-1)!}+\frac{2}{N!}\\+\frac{1}{(N-1)!}-\frac{3}{N!}+\frac{2}{(N+1)!}\end{array}\right]$ $=\frac{1}{5}\left[\frac{1}{0!}-\frac{3}{1!}+\frac{1}{1!}+\frac{2}{N!}+\frac{2}{(N+1)!}-\frac{3}{N!}\right]=\frac{1}{5}\left(\frac{2}{(N+1)!}-\frac{1}{N!}-1\right)$
1(iii)	$\sum_{n=1}^{\infty}\frac{n^2-2n-1}{5(n+1)!}=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{n^2-2n-1}{5(n+1)!}=\lim_{N\to\infty}\left[\frac{1}{5}\left(\frac{2}{(N+1)!}-\frac{1}{N!}-1\right)\right]$ Since $\frac{1}{(N+1)!}\to 0$ & $\frac{1}{N!}\to 0$ when $N\to\infty$, $\sum_{n=1}^{\infty}\frac{n^2-2n-1}{5(n+1)!}$ converges to $-\frac{1}{5}$

Qn	
2(i)	$x=6t^2 \qquad y=\dfrac{2t}{\sqrt{1-t^2}}$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=12t \qquad \dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{2\sqrt{1-t^2}-2t\left(\frac{-2t}{2\sqrt{1-t^2}}\right)}{1-t^2}=\dfrac{2\left(1-t^2\right)+2t^2}{\left(1-t^2\right)\sqrt{1-t^2}}=\dfrac{2}{\left(1-t^2\right)^{3/2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{6t\left(1-t^2\right)^{3/2}}$ The tangent to the curve C has equation $y=\dfrac{1}{6t\left(1-t^2\right)^{3/2}}x$ for some t. $\dfrac{2t}{\sqrt{1-t^2}}=\dfrac{1}{6t\left(1-t^2\right)^{3/2}}\cdot 6t^2 \quad (t\neq 0)$ $2\left(1-t^2\right)=1$ $t^2=\dfrac{1}{2}$ $t=\dfrac{1}{\sqrt{2}}$ since $0<t<1$ Hence the tangent line has equation $y=\dfrac{1}{6\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\right)^{3/2}}x$ $y=\dfrac{2}{3}x$ Alternative method:

Qn

Cartesian equation of curve C:

Sub $t=\sqrt{\frac{x}{6}}$ into $y=\frac{2t}{\sqrt{1-t^2}}$ to get

$$y=\frac{2\sqrt{\frac{x}{6}}}{\sqrt{1-\frac{x}{6}}}=\frac{2\sqrt{x}}{\sqrt{6-x}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cdot\frac{1}{2\sqrt{x}}\cdot\sqrt{6-x}-2\sqrt{x}\cdot\frac{-1}{2\sqrt{6-x}}}{6-x}$$

$$=\frac{6}{(6-x)^{3/2}\sqrt{x}}$$

The required tangent line passes through the point $\left(6t^2,\frac{2t}{\sqrt{1-t^2}}\right)$ for some x.

$$y=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=6t^2}x$$

$$\frac{2t}{\sqrt{1-t^2}}=\frac{6}{\left(6-6t^2\right)^{3/2}\sqrt{6t^2}}\cdot 6t^2 \quad (t\neq 0)$$

$$2=\frac{1}{\left(1-t^2\right)}$$

$$2\left(1-t^2\right)=1$$

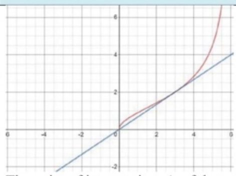
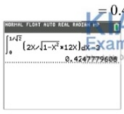
$$t^2=\frac{1}{2}$$

$t=\frac{1}{\sqrt{2}}$ since $0\leq t<1$

Hence the tangent line has equation

$$y=\frac{1}{6\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\right)^{3/2}}x$$

$$y=\frac{2}{3}x$$

Qn	
2(ii)	 The point of intersection, A, of the tangent line and the curve corresponds to $t=\frac{1}{\sqrt{2}}$. Coordinates of point A are $(3,2)$. Area of $R=\int_0^3 y\ \mathrm{d}x-\frac{1}{2}(3)(2)$ $=\int_0^{\frac{1}{\sqrt{2}}} \frac{2t}{\sqrt{1-t^2}}\cdot 12t\mathrm{d}t-\frac{1}{2}(3)(2)$ $=0.425$ (3 s.f.) Alternatively, Area of $R=\int_0^3 y-\frac{2}{3}x\ \mathrm{d}x=\int_0^{\frac{1}{\sqrt{2}}} \frac{2t}{\sqrt{1-t^2}}\cdot 12t\mathrm{d}t-\int_0^3 \frac{2}{3}x\ \mathrm{d}x$ $=0.425$ (3 s.f.) 

Qn	
2(iii)	Sub $t=\sqrt{\frac{x}{6}}$ into $y=\frac{2t}{\sqrt{1-t^2}}$ to get $y=\frac{2\sqrt{\frac{x}{6}}}{\sqrt{1-\frac{x}{6}}}=\frac{2\sqrt{x}}{\sqrt{6-x}}$
2(iv)	Volume, $V=\pi\int_0^3 y^2\ \mathrm{d}x-\frac{1}{3}\pi\left(2^2\right)(3)$ $=\pi\int_0^3 \frac{4x}{6-x}\ \mathrm{d}x-4\pi$ $=\pi\int_0^3 -4+\frac{24}{6-x}\ \mathrm{d}x-4\pi$ $=\pi\left[-4x-24\ln\lvert 6-x\rvert\right]_0^3-4\pi$ $=\pi\left[-12-24\ln 3+24\ln 6\right]-4\pi$ $=\pi\left[24\ln 2\right]-16\pi$ $=\left(24\ln 2-16\right)\pi$ <u>Alternatively,</u> Volume, $V=\pi\int_0^3\left(\frac{2\sqrt{x}}{\sqrt{6-x}}\right)^2-\left(\frac{2}{3}x\right)^2\ \mathrm{d}x$ $=\pi\int_0^3 \frac{4x}{6-x}-\frac{4}{9}x^2\ \mathrm{d}x$ $=\pi\int_0^3 -4+\frac{24}{6-x}-\frac{4}{9}x^2\ \mathrm{d}x$ $=\pi\left[-4x-24\ln\lvert 6-x\rvert-\frac{4}{27}x^3\right]_0^3$ $=\pi\left[\left(-12-24\ln 3-4\right)+24\ln 6\right]$ $=\left(24\ln 2-16\right)\pi$

Qn	
3(i)	Let S_n be the total distance travelled by the ball just before the n-th bounce. Thus $S_n = 10 + 2(10e) + 2(10e^2) + \cdots + 2(10e^{n-1})$ $= 20 + 20e + 20e^2 + \cdots + 20e^{n-1} - 10$ $= \dfrac{20(1-e^n)}{1-e} - 10$ $= \dfrac{10(1+e-2e^n)}{1-e}$
3(ii)	Let d_k be the maximum height of the ball after the k-th bounce. Thus $d_k = 10e^k$. Hence $t_k = 0.90305\sqrt{d_k}$. Thus for $k \in \mathbb{Z}^+$, $\dfrac{t_{k+1}}{t_k} = \dfrac{0.90305\sqrt{d_{k+1}}}{0.90305\sqrt{d_k}}$ $= \dfrac{\sqrt{10e^{k+1}}}{\sqrt{10e^k}} = \sqrt{e}$ Hence t_n is a geometric sequence with common ratio $\sqrt{e}$.
3(iii)	As $n \to \infty$, the ball will come to rest. Thus total distance travelled is $S = \lim_{n\to\infty}\left(\dfrac{10(1+e-2e^n)}{1-e}\right)$ $= \dfrac{10(1+e)}{1-e}$ Total time taken $= 0.5(0.90305)\sqrt{10} + \sum_{n=1}^{\infty} t_n$ $= 1.4278 + \dfrac{0.90305\sqrt{10e}}{1-\sqrt{e}}$ $= 1.43 + \dfrac{2.86\sqrt{e}}{1-\sqrt{e}}$

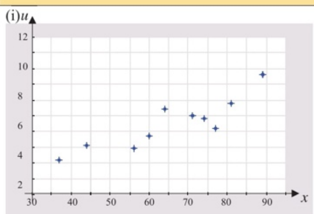
Qn	
4(i)	$\mathbf{n}_{\pi_1} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix} = \begin{pmatrix} 2 \\ -(2a+1) \\ 4 \end{pmatrix}$ $\mathbf{d}_l \cdot \mathbf{n}_{\pi_1} = \begin{pmatrix} 4a \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix}$ $= 8a - 8a - 4 + 4 = 0$ Since $\mathbf{d}_l \perp \mathbf{n}_{\pi_1}$, then l is parallel to π_1
4(ii)	Equation of π_1: $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix} = 10$ Since l is parallel to π_1, we want l to lie inside π_1. $\begin{pmatrix} 3 \\ 0 \\ a \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2a-1 \\ 4 \end{pmatrix} = 10$ $6 + 4a = 10 \quad \Rightarrow \quad a = 1$
4(iii)	Since B lies on the line, required vector is the vector $\overrightarrow{FB}$, where F is the foot of perpendicular from A to π_1. $\overrightarrow{OF} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + k \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ for some $k \in \mathbb{R}$. $\left[\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + k \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 10$ $-19 + 29k = 10 \quad \Rightarrow \quad k = 1$

Qn	
	$\overrightarrow{OF}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ $\overrightarrow{FB}=\begin{pmatrix}3\\0\\1\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$
4(iv)	Let π_2 be the required plane. Point C is the reflection of A in π_1. $\overrightarrow{OC}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}+2\begin{pmatrix}2\\-3\\4\end{pmatrix}=\begin{pmatrix}3\\-3\\6\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}3\\-3\\6\end{pmatrix}-\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}0\\-3\\5\end{pmatrix}$ $\mathbf{n}_{\pi_2}=\begin{pmatrix}4\\4\\1\end{pmatrix}\times\begin{pmatrix}0\\-3\\5\end{pmatrix}=\begin{pmatrix}23\\-20\\-12\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}23\\-20\\-12\end{pmatrix}=\begin{pmatrix}3\\0\\1\end{pmatrix}\cdot\begin{pmatrix}23\\-20\\-12\end{pmatrix}=57$. Thus $23x-20y-12z=57$
4(v)	Maximum value of $\angle ADC=2\times\angle CDF$ $=2\times\cos^{-1}\dfrac{\begin{pmatrix}2\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}23\\-20\\-12\end{pmatrix}}{\left\|\begin{pmatrix}2\\-3\\4\end{pmatrix}\right\|\left\|\begin{pmatrix}23\\-20\\-12\end{pmatrix}\right\|}$ $=2\times\cos^{-1}\dfrac{58}{\sqrt{29}\sqrt{1073}}\approx 2(70.804)$ $=141.6°$ (1dp)

Qn	
5(i)	Number of ways $= \dfrac{9!}{2!2!3!} = 15120$
5(ii)	Number of ways $= \dfrac{7!}{3!} = 840$
5(iii)	Number of ways $= \dfrac{5!}{2!} \cdot {}^6C_3$ $= 1200$
5(iv)	Let the event D be such that the D's are together, the event E be such that the E's are together and S be such that the S's are together. $n(D \cup E \cup S) = n(D) + n(E) + n(S) - n(D \cap E)$ $- n(E \cap S) - n(D \cap S) + n(D \cap E \cap S)$ $= \dfrac{8!}{2!3!} + \dfrac{7!}{2!2!} + \dfrac{8!}{2!3!} - \dfrac{6!}{2!} - \dfrac{6!}{2!} - \dfrac{7!}{3!} + 5!$ $= 6540$ Number of ways $= n(D' \cap E' \cap S')$ $= n(S) - n(D \cup E \cup S)$ $= 15120 - 6540$ $= 8580$

Qn	
6(i)	Let X denotes the number of 1-year old flares that fail to fire successfully, out of the 100, $X \sim \text{B}(100,\ 0.005)$ $\text{P}(X \le 2) = 0.985897 \approx 0.986$
6(ii)	Let Y denotes the number of boxes with a hundred 1-year old flares with at most 2 that fail to fire, out of 50 boxes, ie $Y \sim \text{B}(50,\ 0.985897)$ $\text{P}(Y \le 48) = 0.156856 \approx 0.157$
6(iii)	Let T denotes the number of 10-year old flares that fire successfully, out of the 6, $T \sim \text{B}(6,\ 0.75)$ (a) Required prob = $(1-0.970)\times \text{P}(T \ge 4)$ $= 0.03 \times \left(1 - P(T \le 3)\right)$ $= 0.0249$ (b) P(at least 4 of the 7 flares fire successfully) $= 0.024917 + 0.970 \times \text{P}(T \ge 3)$ $= 0.024917 + 0.970 \times \left(1 - \text{P}(T \le 2)\right)$ $= 0.958$

Qn	
7(i)	Let X be the rv denoting the amount of time taken by a cashier to deal with a randomly chosen customer, ie $X \sim \mathrm{N}(150,\ 45^2)$. $\mathrm{P}(X > 180) = 0.25249 \approx 0.252$
7(ii)	Assume that the time taken to deal with each customer is independent of the other, ie $X_1 + X_2 \sim \mathrm{N}(2\times150, 2\times45^2)$ $\mathrm{P}(X_1 + X_2 < 200) = 0.058051 \approx 0.0581$
7(iii)	Let Y be the rv denoting the amount of time taken by a the second cashier to deal with a randomly chosen customer, ie $Y \sim \mathrm{N}(150,\ 45^2)$. $X_1 + X_2 + X_3 + X_4 \sim \mathrm{N}(4\times150, 4\times45^2)$ and $Y_1 + Y_2 + Y_3 \sim \mathrm{N}(3\times150, 3\times45^2)$ $\mathrm{P}(X_1 + X_2 + X_3 + X_4 < Y_1 + Y_2 + Y_3) = \mathrm{P}(X_1 + X_2 + X_3 + X_4 - (Y_1 + Y_2 + Y_3) < 0)$ Using $X_1 + X_2 + X_3 + X_4 - (Y_1 + Y_2 + Y_3) \sim N(150, 7\times45^2)$ $\mathrm{P}(X_1 + X_2 + X_3 + X_4 - (Y_1 + Y_2 + Y_3) < 0) = 0.10386 \approx 0.104$

Qn	
8(i)	(i) 
8(ii)	Using GC, $r = 0.884$ for the model $u = ax + b$ $u = ae^{bx} \Rightarrow \ln u = bx + \ln a$ Using GC, $r =$ 0.906 for the model $u = ae^{bx}$ Since the value of r is closer to 1 for the 2nd model, $u = ae^{bx}$ is a better model. $\ln u = 0.013633x + 0.94964$ $u = e^{0.013633x+0.94964}$ $u = 2.58e^{0.0136x} = 2.6e^{0.014x}$
8(iii)	$7 = 2.58e^{0.0136x} \Rightarrow x = \dfrac{\ln\left(\dfrac{7}{2.58}\right)}{0.0136} = 73.391 \approx 73$ A patient with urea serum is 7 mmol per litre is approximately 73 years old. Since $r = 0.906$ is close to 1 and 7 is within the data range of urea serum, estimate is reliable.
8(iv) (a)	The product moment correlation coefficient in part **(ii)** will not be changed if the units for the urea serum is given in mmol per decilitre.
8(iv) (b)	$u = 0.258e^{0.0136x}$

Qn	
9(i)	$P(X=2)=\frac{18}{18}\frac{2}{17}\frac{15}{16}\frac{3!}{2!}$ $=\frac{45}{136}$ $P(X=0)=\frac{18}{18}\frac{15}{17}\frac{12}{16}$ $=\frac{45}{68}$ $P(X=3)=\frac{18}{18}\frac{2}{17}\frac{1}{16}$ $=\frac{1}{136}$
9(ii)	$E(X)=\frac{93}{136}$ $E(X^2)=0\times\frac{45}{68}+2^2\times\frac{45}{136}+3^2\times\frac{1}{136}=\frac{189}{136}$ $\text{Var}(X)=\frac{189}{136}-\left(\frac{93}{136}\right)^2$ ≈ 0.922
9(iii)	Since $n=40$ is large, by Central Limit Theorem, $\bar{X}\sim N\left(\frac{93}{136},\frac{0.922}{40}\right)$ approximately $P(\bar{X}>1)=0.0186$

Qn	
9(iv)	Expected winnings $= -\frac{45}{68}a+\frac{45}{136}\left(a+10\right)+\frac{1}{136}(a+10)$ $-\frac{11}{34}a+\frac{115}{34}>0$ $a<\frac{115}{11}$ $a<10.\dot{4}\dot{5}$ The possible amounts will be $1\le a\le 10$ and $a\in\mathbb{Z}$.

Qn	
10(i)	Let X be the thickness of the coating on a randomly chosen computer device. Let μ be the mean thickness of the coating of a computer device. Assume that the standard deviation of the coating of a computer device remains unchanged. To test : $H_0 : \mu = 100$ $H_1 : \mu \neq 100$ Level of Significance: 5% Under H_0, since sample size $n = 50$ is large, by Central Limit Theorem, $Z = \dfrac{\overline{X} - 100}{10/\sqrt{50}} \sim N(0,1)$ approx. Reject H_0 if $p - value \le 0.05$. Calculations: $\overline{x} = 103.4$ $p - value = 0.0162$ Conclusion: Since $p - value < 0.05$, we reject H_0 and conclude that there is significant evidence at 5% level of significance that the process is not in control.
10(ii)	Reject H_0 is $\lvert z_{calc} \rvert \ge 1.960$ For H_0 to be rejected, $\left\lvert \dfrac{\overline{x} - 100}{10/\sqrt{50}} \right\rvert \ge 1.95996$ $\Rightarrow \overline{x} \le 100 - 1.95996\left(\dfrac{10}{\sqrt{50}}\right)$ or $\overline{x} \ge 100 + 1.95996\left(\dfrac{10}{\sqrt{50}}\right)$ $\Rightarrow \overline{x} \le 97.228$ or $\overline{x} \ge 102.772$ Thus the required range of values of $\overline{x}$ is $0 < \overline{x} \le 97.2$ or $\overline{x} \ge 102.8$.

Qn	
10(iii)	$\overline{y}=\frac{4164}{40}=104.1$ $\Sigma(y-100)=4164-4000=164$ $s^2=\frac{1}{39}\left[\Sigma(y-100)^2-\frac{\left(\Sigma(y-100)\right)^2}{40}\right]$ $=\frac{1}{39}\left[9447-\frac{164^2}{40}\right]$ $=\frac{43873}{195}=224.9897$
10(iv)	The standard deviation may have changed due to the wear out of mechanical parts as well.
10(v)	To test : $H_0:\mu=100$, $H_1:\mu\neq 100$ Level of Significance: 4% Under H_0, since sample size $n=40$ is large, by Central Limit Theorem, $Z=\frac{\overline{Y}-100}{S/\sqrt{40}}\sim N(0,1)$ approx. Reject H_0 if $p-value \le 0.04$. Calculations: $\overline{x}=104.1, s^2=224.9897$ $p-value=0.0839$ Conclusion: Since $p-value>0.04$, we do not reject H_0 and conclude that there is insignificant evidence at 4% level of significance that the process is not in control.