8 The function f is defined by $\mathrm{f}: x \mapsto \sqrt{a^2 - \frac{(x-a)^2}{4}}$ for $x \in \mathbb{R}$, $-a \le x \le 3a$, where a is a positive constant.

(i) Sketch the graph of $y = \mathrm{f}(x)$, giving the coordinates of any stationary points and the points where the graph meets the axes. [2]

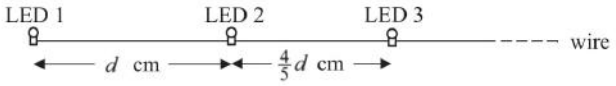
(ii) If the domain of f is further restricted to $-a \le x \le k$, state the greatest value of k for which the function f^{-1} exists. [1]

(iii) Using the restricted domain found in part **(ii)**, find f^{-1} in similar form. [3]

The function g is defined by $\mathrm{g}(x) = \mathrm{f}\left(\frac{3}{2}x\right)$ for $x \in \mathbb{R}$, $-\frac{2}{3}a \le x \le 2a$.

(iv) Explain why the composite function gf exists and find the range of gf. [3]

9 A company produces festive decorative Light Emitting Diode (LED) string lights, where micro LEDs are placed at intervals along a thin wire. In a particular design, the first LED (LED 1) is placed on one end of a wire with the second LED (LED 2) placed at a distance of d cm from LED 1, and each subsequent LED is placed at a distance $\frac{4}{5}$ times the preceding distance as shown.



(i) If the distance between LED 8 and LED 9 is 56.2 cm, find the value of d correct to 1 decimal place. [2]

(ii) Find the theoretical maximum length of the wire, giving your answer in centimetres correct to 1 decimal place. [2]

The LEDs consisting of three colours red, orange and yellow, are arranged in that order in a repeated manner, that is, LED 1 is red, LED 2 is orange, LED 3 is yellow, LED 4 is red, LED 5 is orange, LED 6 is yellow, and so on.

(iii) Find the colour of the LED nearest to a point on the wire 12.9 m from LED 1. [3]

(iv) If the minimum distance between any two consecutive LEDs is 1 cm so that they can be mounted on the wire, find the colour of the last LED. [3]

2019 H2 Math Prelim - HCI

1 The number of units, $D(x)$ of a particular product that people are willing to purchase per week in city A at a price $\$x$ is given by the function $D(x)=\dfrac{40320}{\mathrm{g}(x)}$, where $\mathrm{g}(x)$ is a quadratic polynomial in x. The following table shows the number of units people are willing to purchase at different prices.

x	5	8	10
$D(x)$	384	224	168

Find the number of units of the product that people are willing to purchase at a price of \$18. [4]

2 It is given that $p_n=\ln\dfrac{1+x^n}{1+x^{n+1}}$, where $-1<x<1$ and n is a positive integer.

(i) Find $\sum_{n=1}^{N} p_n$, giving your answer in terms of N and x. [3]

(ii) Hence find the sum to infinity of the series in part **(i)** in terms of x. [2]

3 In the triangle ABC, $AB=1$, $AC=2$ and angle $ABC=\left(\dfrac{\pi}{2}-x\right)$ radians. Given that x is sufficiently small for x^3 and higher powers of x to be ignored, show that $BC\approx p+qx+rx^2$, where p, q, r are constants to be determined in exact form. [5]

4 A part of a hyperbola has equation given by $\mathrm{f}(x)=\sqrt{\dfrac{(x+5)^2}{36}-1}$, $x\in D$, where $D\subseteq\mathbb{R}$.

(i) State the largest possible set D. [1]

(ii) State the equations of the asymptotes of $y=\mathrm{f}(x)$. [1]

(iii) Sketch the graph of $y=\mathrm{f}(x)$ for D in part **(i)**, showing clearly all the features of the curve. [2]

(iv) On separate diagrams, sketch the graphs of $y=\dfrac{1}{\mathrm{f}(x)}$ and $y=\mathrm{f}'(x)$, showing clearly all the features of the curves. [4]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OB produced such that $\overrightarrow{OC} = \lambda\overrightarrow{OB}$ where $\lambda > 1$. Point D is such that $OCDA$ is a parallelogram. Point M lies on AD, between A and D, such that $AM : MD = 1 : 2$. Point N lies on OC, between O and C, such that $ON : NC = 4 : 3$.

(i) Find the position vectors of M and N, in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. [2]

(ii) Show that the area of triangle OMD is $k\lambda|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be determined. [4]

(iii) The vector $\mathbf{p}$ is a unit vector in the direction of $\overrightarrow{OD}$. Give a geometrical meaning of $|\mathbf{p}.\mathbf{a}|$. [1]

6 A curve C is defined by the parametric equations

$$x = 25\sin^2 t, \quad y = 2\cos t$$

where $0 \le t \le \pi$.

(i) Find $\dfrac{dy}{dx}$. [2]

(ii) The tangent to C at the point where $t = \dfrac{\pi}{4}$ cuts the x-axis at P and the y-axis at Q, find the exact area of the triangle OPQ, where O is the origin. [4]

(iii) State the equation of the normal to C where the normal is parallel to the x-axis. [1]

7 **(a)** **(i)** Find $\dfrac{d}{dx}\left(2e^{\cos\frac{x}{2}}\right)$. [1]

(ii) Hence find $\displaystyle\int \frac{1}{2}\sin x\, e^{\cos\frac{x}{2}}\, dx$. [4]

(b) Using the substitution $u = 1 - e^x$, find $\displaystyle\int \frac{1}{1-e^x}\, dx$. [4]

10 **(a)** Solve the simultaneous equations $v + \mathrm{i}u = 2$ and $av - 2u = 3\mathrm{i}$, where a is a real constant. Simplify your answers to cartesian form $x + \mathrm{i}y$, where x and y are in terms of a. [4]

(b) It is given that $(x + k)$ is a factor of the equation,

$$bx^3 + (12b + \mathrm{i})x^2 + (b + 12\mathrm{i})x + 12b = 0,$$

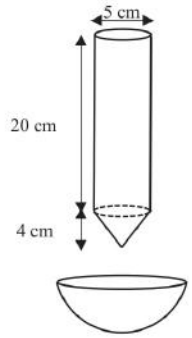
where k and b are non-zero real constants.

(i) Find the value of k. [2]

(ii) Show that the roots of the equation $bx^2 + \mathrm{i}x + b = 0$ are purely imaginary. [2]

(iii) Hence express $\mathrm{f}(x) = bx^3 + (12b + \mathrm{i})x^2 + (b + 12\mathrm{i})x + 12b$ as a product of three linear factors, leaving your answers in terms of b. [2]

11 A container is made up of a cylinder and an inverted right circular cone as shown in the diagram below. The height and the diameter of the cylinder are 20 cm and 5 cm respectively. The height of the cone is 4 cm. An external device ensures liquid flows out through a small hole at the vertex of the cone into a bowl below at a constant rate of 18 cm^3 per minute. The depth of the liquid and the radius of the liquid surface area in the container at the time t minutes are x cm and r cm respectively. The container is full of liquid initially.

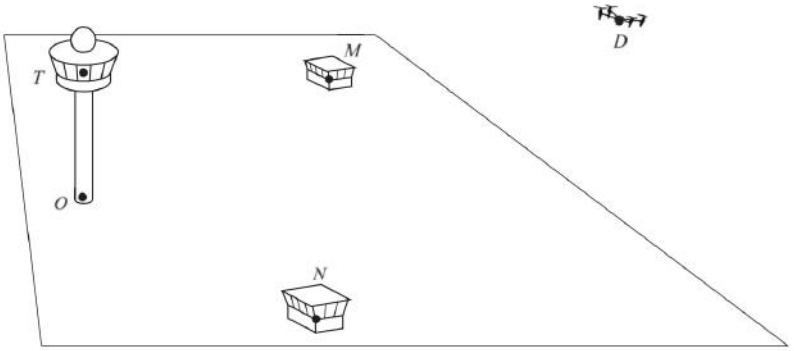


(a) When $x > 4$, find the rate of change of the depth of the liquid in the container. [4]

(b) Find the rate of change of r when $x = 2$. [5]

(c) The bowl as shown in the diagram in part **(a)** is generated by rotating part of the curve $\frac{x^2}{225} + \frac{y^2}{100} = 1$ which is below the x-axis through π radians about the y-axis. Assuming the bowl has negligible thickness, find the volume of the empty space in the bowl when the liquid has completely flowed from the container into the bowl, giving your answer correct to 3 decimal places. [3]

12 At an airport, an air traffic control room T is located in a vertical air traffic control tower, 70 m above ground level. Let $O(0,0,0)$ be the foot of the air traffic control tower and all points (x,y,z) are defined relative to O where the units are in kilometres. Two observation posts at the points $M(0.8,0.6,0)$ and $N(0.4,-0.9,0)$ are located within the perimeters of the airport as shown.



An air traffic controller on duty at T spots an errant drone in the vicinity of the airport. The two observation posts at M and N are alerted immediately. A laser rangefinder at M directs a laser beam in the direction $\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$ at the errant drone to determine D, the position of the errant drone. The position D is confirmed using another laser beam from N, which passes through the point $(0.8, 0.75, 0.3)$, directed at the errant drone.

(i) Show that D has coordinates $(0.56,-0.24,0.12)$. [4]

A Drone Catcher, an anti-drone drone which uses a net to trap and capture errant drones, is deployed instantly from O and flies in a straight line directly to I intercept the errant drone.

(ii) Find the acute angle between the flight path of the Drone Catcher and the horizontal ground. [2]

At the same time, a Jammer Gun, which emits a signal to jam the control signals of the errant drone, is fired at the errant drone. The Jammer Gun is located at a point G on the plane p containing the points T, M and N.

(iii) Show that the equation of p is $\mathbf{r} \cdot \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = -6.72$. [3]

It is also known that the Jammer Gun is at the foot of the perpendicular from the errant drone to plane p.

(iv) Find the coordinates of G. [3]

(v) Hence, or otherwise, find the distance GD in metres. [2]

Section A: Pure Mathematics [40 marks]

1 On the same axes, sketch the graphs of $y = 2(x-a)^2$ and $y = 3a|x-a|$, where a is a positive constant, showing clearly all axial intercepts. [2]

(i) Solve the inequality $2(x-a)^2 \geq 3a|x-a|$. [4]

(ii) Hence solve $2\left(x-\frac{a}{2}\right)^2 \geq 3a\left|x-\frac{a}{2}\right|$. [2]

2 It is given that $y = \frac{e^{\sin x}}{\sqrt{1+2x}}$.

(i) Show that $\frac{1}{y}\frac{dy}{dx} + \frac{1}{1+2x} = \cos x$. [2]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin series for y in ascending powers of x, up to and including the term in x^3. [5]

(iii) Use your result from part **(ii)** to approximate the value of $\int_0^1 \frac{e^{\sin x}}{\sqrt{1+2x}}\,dx$. Explain why this approximation obtained is not good. [2]

(iv) Deduce the Maclaurin series for $\frac{1}{e^{\sin x}\sqrt{1-2x}}$ in ascending powers of x, up to and including the term in x^3. [1]

3 The complex numbers p and q are given by $\frac{a}{1+\sqrt{3}i}$ and $-\frac{a}{2}i$ respectively, where a is a positive real constant.

(i) Find the modulus and argument of p. [2]

(ii) Illustrate on an Argand diagram, the points P, Q and R representing the complex numbers p, q and $p+q$ respectively. State the shape of $OPRQ$.
Hence, find the argument of $p+q$ in terms of π and the modulus of $p+q$ in exact trigonometrical form. [6]

(iii) Find the smallest positive integer n such that $(p+q)^n$ is purely imaginary. [2]

4 **(a)** When studying a colony of bugs, a scientist found that the birth rate of the bugs is inversely proportional to its population and the death rate is proportional to its population. The population of the bugs (in thousands) at time t days after they were first observed is denoted by P. It was found that when the population is 2000, it remains constant.

(i) Assuming that P and t are continuous variables, show that $\frac{\mathrm{d}P}{\mathrm{d}t} = k\left(\frac{4}{P} - P\right)$, where k is a constant. [3]

(ii) Given that the initial population of the bugs was 4000, and that the population was decreasing at the rate of 3000 per day at that instant, find P in terms of t. [4]

(iii) Sketch the graph of P against t, giving the equation of any asymptote(s). State what happens to the population of the bugs in the long run. [2]

(b) Another population of bugs, N (in thousands) in time t days can be modelled by the differential equation $\frac{\mathrm{d}N}{\mathrm{d}t} = 4 + \frac{N}{t}$ for $t \geq 1$. Using the substitution $u = \frac{N}{t}$, solve this equation, given that the population was 1000 when $t = 1$. [3]

Section B: Statistics [60 marks]

5 The daily rainfall in a town follows a normal distribution with mean μ mm and standard deviation σ mm. Assume that the rainfall each day is independent of the rainfall on other days. It is given that there is a 10% chance that the rainfall on a randomly chosen day exceeds 9.8 mm, and there is a 10% chance that the mean daily rainfall in a randomly chosen 7-day week exceeds 8.2 mm.

(i) Show that $\sigma = 2.01$, correct to 2 decimal places. [4]

(ii) Find the maximum value of k such that there is a chance of at least 10% that the mean daily rainfall in a randomly chosen 30-day month exceeds k mm. Give your answer correct to 1 decimal place. [2]

6 Miss Tan carried out an investigation on whether there is a correlation between the amount of time spent on social media and exam scores. The average amount of time spent per month on social media, x hours, and the final exam score, y marks, of 6 randomly selected students from HCI were recorded. The data is shown below.

x	80	84	70	74	58	48
y	44	40	49	45	58	82

(i) Draw a scatter diagram to illustrate the data. [2]

(ii) It is found that the inclusion of a 7th point (x_7, y_7) will not affect the product moment correlation coefficient for the data. Find a possible point (x_7, y_7). [1]

Omit the 7th point (x_7, y_7) for the rest of this question.

(iii) State, with reason, which of the following equations, where a and b are constants, provides the most appropriate model for the relationship between x and y.

(A) $y = a + bx^2$,

(B) $e^y = ax^b$,

(C) $y = a + b\sqrt{x}$. [3]

(iv) Using the model chosen in part **(iii)**, estimate the score of a student who spent an average of 60 hours per month on social media, giving your answer correct to the nearest whole number. [2]

(v) Sam spends an average of 4 hours a day on social media. Assuming a 30-day month, suggest whether it is still reasonable to use the model in part **(iii)** to estimate his score. [1]

7 A cafe sells sandwiches in 2 sizes, "footlong" and "6-inch". The lengths in inches of "footlong" loaves have the distribution $N(12.2, 0.04)$ and the lengths in inches of "6-inch" loaves have the distribution $N(6.1, 0.02)$.

(i) Is a randomly chosen "footlong" loaf more likely to be less than 12 inches in length or a randomly chosen "6-inch" loaf more likely to be less than 6 inches in length? [2]

(ii) Find the probability that two randomly chosen "6-inch" loaves have total length more than one randomly chosen "footlong" loaf. [2]

Sue buys a "6-inch" sandwich 3 times a week.

(iii) Find the probability that Sue gets at most one sandwich that is less than 6 inches in length in a randomly chosen week. [2]

(iv) Given that Sue gets more than four sandwiches that are less than 6 inches in length in a randomly chosen 4-week period, find the probability that she gets exactly one such sandwich in the first week. [3]

8 The individual letters of the word PARALLEL are printed on identical cards and arranged in a straight line.

(a) Find the number of arrangements such that

(i) there are no restrictions, [1]

(ii) no L is next to any other L, [2]

(iii) the arrangements start and end with a consonant and all the vowels are together. [3]

(b) The cards are now placed in a bag and Tom draws the cards randomly from the bag one at a time.

(i) 4 cards are drawn without replacement. Find the probability that there is at least one vowel drawn. [2]

(ii) Tom decides to record the letter of the card drawn, on a piece of paper. If the letter on the card drawn is a vowel, Tom will put the drawn card back into the bag and continue with the next draw.

If the letter on the card drawn is a consonant, Tom will remove the card from subsequent draws. Find the probability that Tom records more consonants than vowels at the end of 3 draws. [3]

9 A company purchased a machine to pack shower gel into its bottles. The expected mean volume of shower gel in a bottle is 950 ml.

(a) The floor supervisor believes that the machine is packing less amount of shower gel than expected. A random sample of 80 bottles is taken and the data is as follows:

Volume of shower gel in a bottle (correct to nearest ml)	948	949	950	951	952	953	955
Number of bottles	9	22	36	6	4	1	2

(i) Find unbiased estimates of the population mean and variance, giving your answers correct to 2 decimal places. [2]

(ii) Write down the appropriate hypotheses to test the floor supervisor's belief. You should define any symbols used. [2]

(iii) Using the given data, find the p-value of the test. State what is meant by this p-value in the context of this question. [2]

(iv) It was concluded at $\alpha\%$ level of significance that the machine is indeed packing less amount of shower gel than expected. State the set of values of α. [1]

(b) Due to a change in marketing policy, the machine is being recalibrated to pack smaller bottles of shower gel with mean volume of 250 ml. The volume of a recalibrated bottle of shower gel is denoted by Y ml. A random sample of 50 bottles of y ml each is taken and the data obtained is summarised by:

$$\sum(y-250)=-25, \qquad \sum(y-250)^2=k.$$

Another test was conducted at the 1% significance level. The test concluded that the machine had been calibrated incorrectly. Find the range of values of k, correct to 1 decimal place. [4]

(c) Explain why there is no need for the floor supervisor to know anything about the population distribution of the volume of shower gel in a bottle for both parts **(a)** and **(b)**. [1]

10 In a game with a 4-sided fair die numbered 1 to 4 on each face, the score for a throw is the number on the bottom face of the die. A player gets to choose either option A or option B.

Option A: The player rolls the die once. The score x is the amount of money \$$x$ that the player wins.

Option B: The player rolls the die twice. The first score is x and the second score is y. If $y > x$, the player wins \$$2xy$, but if $y < x$, the player loses \$$(x - y)$. Otherwise, he neither wins nor loses any money.

(i) Find the expected amounts won by a player in one game when playing option A and when playing option B. Show that option B is a better option. [5]

(ii) Suggest why a risk averse player would still choose option A. [1]

(iii) Show that the variance of the amount won by a player in one game when playing option A is 1.25. [2]

In a competition, Abel and Benson each play the game 50 times. Abel chooses option A and Benson chooses option B.

It is given that the variance of the amount won by a player in one game when playing option B is $\frac{887}{16}$.

(iv) Find the distributions of the total amounts won by Abel and Benson respectively in the competition. [2]

(v) Show that the probability of the total amount won by Abel exceeding the total amount won by Benson in the competition is approximately 0.120. [3]

ANNEX

H2 MA 2019 JC2 Prelim (Paper 1 and Paper 2)

***Filename**: Change* SCHOOL *to your school name, e.g. NYJC*

Paper 1

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 1	ANSWERS (Exclude graphs and text answers)
1	Equations & Inequalities	72 units
2	Sigma Notation & MOD	(i) $\ln\dfrac{1+x}{1+x^{N+1}}$ (ii) $\ln(1+x)$
3	Maclaurin & Binomial Series	$p=\sqrt{3}, q=1, r=\dfrac{1}{2\sqrt{3}}$
4	Graphs & Transformations	(i) $D=(-\infty,-1]\cup[1,\infty)$ (ii) $y=\pm\dfrac{x+5}{6}$
5	Vectors	(i) $\overrightarrow{OM}=\dfrac{1}{3}(3\mathbf{a}+\lambda\mathbf{b}), \overrightarrow{ON}=\dfrac{4}{7}\lambda\mathbf{b}$ (ii) $k=\dfrac{1}{3}$
6	Differentiation & Applications	(i) $\dfrac{dy}{dx}=\dfrac{-1}{25\cos t}$ (ii) $\dfrac{225}{4\sqrt{2}}$ (iii) $y=0$
7	Integration & Applications	(a)(i) $-\sin\dfrac{x}{2}\left(e^{\cos\frac{x}{2}}\right)$ (ii) $-2\cos\dfrac{x}{2}\left(e^{\cos\frac{x}{2}}\right)+2e^{\cos\frac{x}{2}}+C$ (b) $-\ln\lvert 1-e^{x}\rvert+x+C$
8	Functions	(ii) greatest value of k is a. (iii) $f^{-1}: x\mapsto a-2\sqrt{a^2-x^2},\ x\in\mathbb{R},\ 0\le x\le a$ (iv) $R_{gf}=\left[\dfrac{\sqrt{3}}{2}a, a\right]$
9	APGP	(i) $d=268.0$ (ii) 1339.9 cm (iii) red

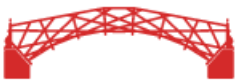
		(iv) yellow
10	Complex Numbers	(a) $u=\frac{a}{4+a^2}-\frac{6+2a^2}{4+a^2}\mathrm{i}, v=\frac{2}{4+a^2}-\frac{a}{4+a^2}\mathrm{i}$ (b)(i) $k=12$ (iii) $b(x+12)(x+\frac{1+\sqrt{1+4b^2}}{2b}\mathrm{i})(x+\frac{1-\sqrt{1+4b^2}}{2b}\mathrm{i})$
11	Differentiation & Applications	(a) -0.916 cm/min (b) -2.29 cm/min (c) 4293.510 cm^3
12	Vectors	(ii) $11.1°$ (iv) $(0.547,-0.237,0.00325)$ (v) 117 m
13	H2 Prelim P1 Q13 Topic	
14	H2 Prelim P1 Q14 Topic	

Paper 2

Select topic from the dropdown list. If the question consists of multiple topics, choose 1 topic.

QN	TOPIC (H2) Paper 2	ANSWERS (Exclude graphs and text answers)
1	Equations & Inequalities	(i) $x\le-\frac{a}{2}$ or $x=a$ or $x\ge\frac{5a}{2}$ (ii) $x\le-a$ or $x=\frac{a}{2}$ or $x\ge 2a$
2	Maclaurin & Binomial Series	(ii) $y=1+x^2-\frac{3}{2}x^3+\ldots$ (iii) 0.985 (iv) $\frac{1}{e^{\sin x}\sqrt{1-2x}}\approx 1+x^2+\frac{3}{2}x^3$
3	Complex Numbers	(i) $\frac{a}{2}; -\frac{\pi}{3}$ (ii) $-\frac{5\pi}{12}; a\cos\frac{\pi}{12}$ (iii) 6
4	Differential Equations	(a) (ii) $P=2\sqrt{1+3e^{-2t}}$ (iii) The population of the bugs will decrease and approach 2000 in the long run. (b) $N=4t\ln t+t$
5	Normal Distribution	(ii) 7.6
6	Correlation & Regression	(ii) $(69,53)$ (iii) Since $\lvert r\rvert=0.96785$ for Model B is nearest to 1, Model B is the most accurate model (iv) 62 marks (v) unreasonable as 120 is not within data range

		where $48 \le x \le 84$
7	Normal Distribution	(i) a "6-inch" loaf more likely to be less than 6 inches (ii) $\frac{1}{2}$ (iii) 0.855 (iv) 0.446
8	PnC & Probability	(a) (i) 3360 (ii) 1200 (iii) 240 (b) (i) $\frac{13}{14}$ (ii) 0.644
9	Hypothesis Testing	(a) (i) 949.84 ; 1,73 (ii) H_0: $\mu = 950$ H_1: $\mu < 950$ (iii) 0.135 (iv) $\{\alpha \in \mathbb{R} : 13.5 \le \alpha \le 100\}$ (b) $12.5 \le k \le 104.8$ (c) There is no need for the floor supervisor to assume the volume of shower gel follow a normal distribution as the sample sizes in both part (a) and (b) are large, the sample mean volume of shower gel can be approximated to follow a normal distribution by Central Limit Theorem.
10	DRV	(i) \$2.50; \$3.75 (ii) Option A is a "sure win" option where the player would definitely gain a positive amount in all cases, whereas option B has a risk of losing money in some cases. (iv) $A \sim N\left(125, \frac{125}{2}\right)$; $B \sim N\left(\frac{375}{2}, \frac{22175}{8}\right)$
11	H2 Prelim P2 Q11 Topic	
12	H2 Prelim P2 Q12 Topic	
13	H2 Prelim P2 Q13 Topic	
14	H2 Prelim P2 Q14 Topic	



Dr.Kenny Education

2019 C2 H2 Prelim P1 Markers Commen

Qn	Solutions	C
1	$D(x)=\dfrac{40320}{g(x)}$ $\Rightarrow g(x)=\dfrac{40320}{D(x)}$ $ax^2+bx+c=\dfrac{40320}{D(x)}$ Given $5^2a+5b+c=\dfrac{40320}{384}=105$ -- (1) $8^2a+8b+c=\dfrac{40320}{224}=180$ -- (2) $10^2a+10b+c=\dfrac{40320}{168}=240$ -- (3) Using GC, $a=1,\ b=12,\ c=20$ When $x=18$, $D(18)=\dfrac{40320}{18^2+12(18)+20}=72$ units	G w A g(5^2 of 5^2 A de in pc
2i	Method 1: (method of differences) $\sum_{n=1}^{N}p_n=\sum_{n=1}^{N}\ln\dfrac{1+x^n}{1+x^{n+1}}$ $=\sum_{n=1}^{N}\left(\ln(1+x^n)-\ln(1+x^{n+1})\right)$ $=\ln(1+x)-\ln(1+x^2)$ $+\ln(1+x^2)-\ln(1+x^3)$ $+\ln(1+x^3)-\ln(1+x^4)$ $\vdots$ $+\ln(1+x^N)-\ln(1+x^{N+1})$ $=\ln(1+x)-\ln(1+x^{N+1})$ $=\ln\dfrac{1+x}{1+x^{N+1}}$	Tl dc dc su di cl th Sc en lir co se A fir ln of

Method 2: (using property of logarithm)

$$\sum_{n=1}^{N} p_n = \sum_{n=1}^{N} \ln \frac{1+x^n}{1+x^{n+1}}$$

$$= \ln \frac{1+x}{1+x^2} + \ln \frac{1+x^2}{1+x^3} + \ln \frac{1+x^3}{1+x^4} + \ldots + \ln \frac{1+x^N}{1+x^{N+1}}$$

$$= \ln \frac{1+x}{1+x^2} \cdot \frac{1+x^2}{1+x^3} \cdot \frac{1+x^3}{1+x^4} \cdot \ldots \cdot \frac{1+x^N}{1+x^{N+1}}$$

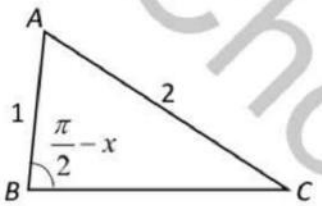
$$= \ln \frac{1+x}{1+x^{N+1}}$$

2ii Since $-1 < x < 1$, as $N \to \infty$, $x^{N+1} \to 0$.

$$\therefore \sum_{n=1}^{\infty} p_n = \ln \frac{1+x}{1+0}$$

$$= \ln(1+x)$$

3



By Cosine Rule,

$$\cos\left(\frac{\pi}{2} - x\right) \quad 2^2 = 1^2 + (BC)^2 - 2(BC)\cos\left(\frac{\pi}{2} - x\right)$$

$$4 = 1 + (BC)^2 - 2(BC)\sin x$$

$$(BC)^2 - 2(BC)\sin x - 3 = 0$$

$$BC = \frac{2\sin x \pm \sqrt{4\sin^2 x + 12}}{2}$$

$$= \sin x \pm \sqrt{\sin^2 x + 3}$$

$$\approx x \pm \sqrt{x^2 + 3}$$

$$= x + \sqrt{x^2 + 3} \quad , \text{ since } BC > 0$$

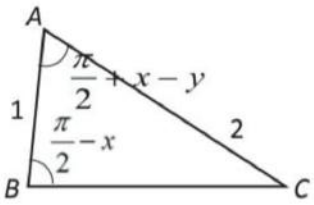
$$= x + \sqrt{3}\left(1 + \frac{x^2}{3}\right)^{1/2}$$

$$= x + \sqrt{3}\left[1 + \frac{1}{2}\frac{x^2}{3} + \ldots\right]$$

$$\approx \sqrt{3} + x + \frac{x^2}{2\sqrt{3}}$$

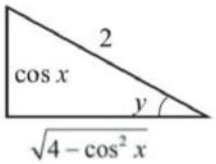
$$p = \sqrt{3}, q = 1, r = \frac{1}{2\sqrt{3}}$$

Alternative Solution



By sine rule,

$$\frac{BC}{\sin\left(\frac{\pi}{2}+x-y\right)}=\frac{2}{\sin\left(\frac{\pi}{2}-x\right)}=\frac{1}{\sin y}$$



$$\Rightarrow \frac{BC}{\cos(y-x)}=\frac{2}{\cos x}=\frac{1}{\sin y}$$

$$\Rightarrow BC=\frac{2\cos(y-x)}{\cos x}$$

$$\Rightarrow BC=\frac{2(\cos y\cos x+\sin y\sin x)}{\cos x}$$

$$\Rightarrow BC=\frac{2(\cos y\cos x+\sin y\sin x)}{\cos x}$$

$$\Rightarrow BC=\frac{2\left(\frac{\sqrt{4-\cos^2 x}}{2}\cos x+\frac{\cos x}{2}\sin x\right)}{\cos x}$$

$$\Rightarrow BC=\frac{\sqrt{4-\cos^2 x}\cos x+\cos x\sin x}{\cos x}$$

$$\Rightarrow BC=\sqrt{4-\cos^2 x}+\sin x$$

$$\Rightarrow BC=\sqrt{3+\sin^2 x}+\sin x$$

4i Sketch the graph using GC, remember to set window appropriately.

OR

Notice that it is the upper half of a hyperbola.

$$y=\sqrt{\frac{(x+5)^2}{36}-1}$$

$$\frac{(x+5)^2}{6^2}-y^2=1$$

We have $D=(-\infty,-11]\cup[1,\infty)$.

ii

Let $y = \sqrt{\frac{(x+5)^2}{36} - 1}$

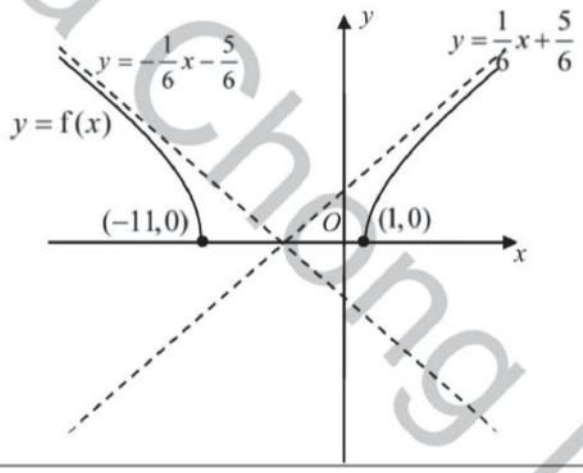
$\frac{(x+5)^2}{6^2} - y^2 = 1$

To find asymptotes:

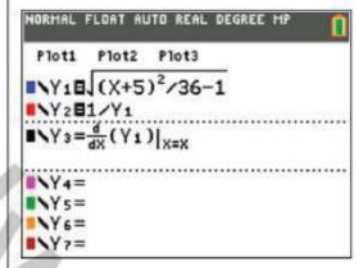
$y^2 = \frac{(x+5)^2}{6^2}$

$y = \pm\frac{x+5}{6}$

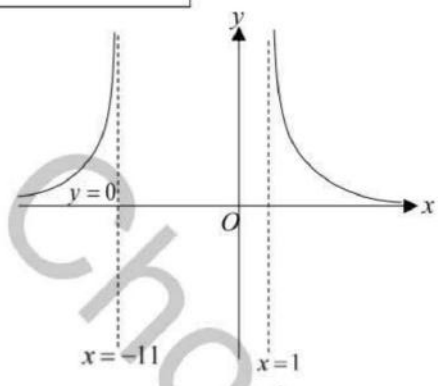
iii



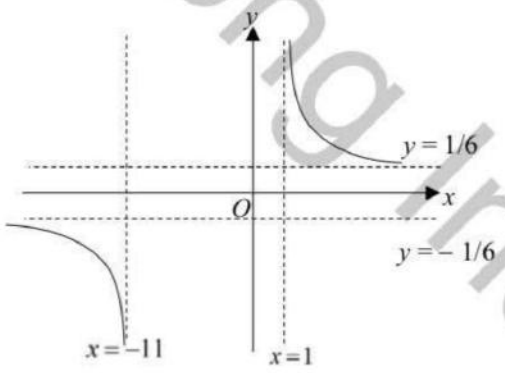
iv Using GC to help if students is not sure



$y = \dfrac{1}{\mathrm{f}(x)}$



$y = \mathrm{f}'(x)$



5i

$\overrightarrow{OC} = \lambda \mathbf{b}$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OC} = a + \lambda \mathbf{b}$

$\overrightarrow{ON} = \frac{4}{7}\lambda \mathbf{b}$

$\overrightarrow{OM} = \frac{\overrightarrow{OD} + 2\overrightarrow{OA}}{3}$

$= \frac{\mathbf{a} + \lambda \mathbf{b} + 2\mathbf{a}}{3}$

$= \frac{1}{3}(3\mathbf{a} + \lambda \mathbf{b})$

ii

Area of triangle $OMD = \frac{1}{2}\left|\overrightarrow{OM} \times \overrightarrow{OD}\right|$

$= \frac{1}{2}\left|\frac{1}{3}(\lambda \mathbf{b} + 3\mathbf{a}) \times (\lambda \mathbf{b} + \mathbf{a})\right|$

$= \frac{1}{6}\left|\lambda^2 \mathbf{b} \times \mathbf{b} + \lambda \mathbf{b} \times \mathbf{a} + 3\lambda \mathbf{a} \times \mathbf{b} + 3\mathbf{a} \times \mathbf{a}\right|$

$= \frac{1}{6}\left|-\lambda \mathbf{a} \times \mathbf{b} + 3\lambda \mathbf{a} \times \mathbf{b}\right| \quad (\because \mathbf{b} \times \mathbf{b} = \mathbf{0} = \mathbf{a} \times \mathbf{a})$

$= \frac{1}{3}\lambda \left|\mathbf{a} \times \mathbf{b}\right|$

iii

$|\mathbf{p}.\mathbf{a}|$ is the length of projection of $\overrightarrow{OA}$ on $\overrightarrow{OD}$

6i	$\frac{dx}{dt} = 50\sin t\cos t$, $\frac{dy}{dt} = -2\sin t$ $\frac{dy}{dx} = \frac{-1}{25\cos t}$	
ii	When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-\sqrt{2}}{25}$, $x = \frac{25}{2}$, $y = \sqrt{2}$ $y = -\frac{\sqrt{2}}{25}x + c$ $c = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ Equation of tangent: $y = -\frac{\sqrt{2}}{25}x + \frac{3}{\sqrt{2}}$ When $x = 0$, $y = \frac{3}{\sqrt{2}}$ When $y = 0$, $x = \frac{75}{2}$ Area of triangle $OPQ = \frac{1}{2}\left(\frac{75}{2}\right)\left(\frac{3}{\sqrt{2}}\right) = \frac{75}{4}\left(\frac{3}{\sqrt{2}}\right) = \frac{225}{4\sqrt{2}}$	

iii

$y = 0$

7ai

Let $y = 2e^{\cos\frac{x}{2}}$

$$\frac{dy}{dx} = -\frac{1}{2}\sin\frac{x}{2}\left(2e^{\cos\frac{x}{2}}\right)$$

$$= -\sin\frac{x}{2}\left(e^{\cos\frac{x}{2}}\right)$$

aii

$$\int \frac{1}{2}\sin x\left(e^{\cos\frac{x}{2}}\right)dx$$

$$= \int \frac{1}{2}\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\left(e^{\cos\frac{x}{2}}\right)dx$$

$$= -\int\left(\cos\frac{x}{2}\right)\left(-\sin\frac{x}{2}e^{\cos\frac{x}{2}}\right)dx$$

$$= \left(-\cos\frac{x}{2}\right)\left(2e^{\cos\frac{x}{2}}\right) - \int\left(\frac{1}{2}\sin\frac{x}{2}\right)\left(2e^{\cos\frac{x}{2}}\right)dx$$

$$= -2\cos\frac{x}{2}\left(e^{\cos\frac{x}{2}}\right) + 2e^{\cos\frac{x}{2}} + C$$

b

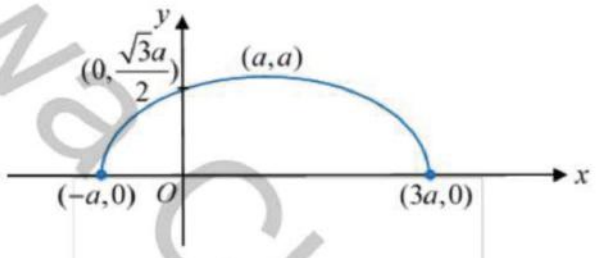
$$\int \frac{1}{1-e^x}\,dx$$

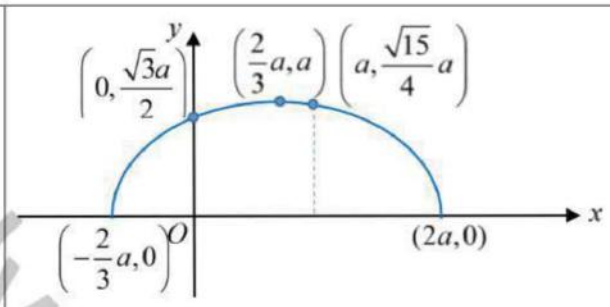
Let $u = 1 - e^x \Rightarrow \frac{du}{dx} = -e^x = u - 1 \Rightarrow \frac{dx}{du} = \frac{1}{u-1}$

$$= \int \frac{1}{u(u-1)}\,du$$

$$= \int -\frac{1}{u} + \frac{1}{u-1}\,du$$

$$= -\ln|u| + \ln|u-1| + C$$

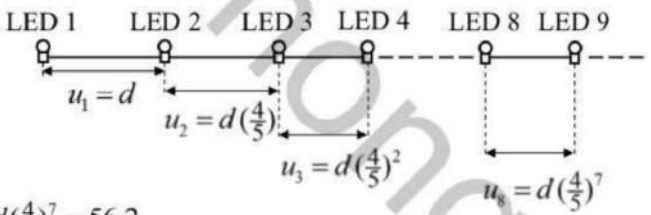
	$= -\ln\left\|1-e^x\right\| + \ln\left\|-e^x\right\| + C$ $= -\ln\left\|1-e^x\right\| + x + C$
8i	
ii	For f^{-1} to exist, f must be one-one. i.e. every horizontal line $y=h$, $h\in\mathbb{R}$ can only cut the graph of $y=f(x)$ at most once. Hence, the greatest value of k is a.
iii	Let $y=\sqrt{a^2-\frac{(x-a)^2}{4}}$ $\Rightarrow y^2 = a^2 - \frac{(x-a)^2}{4}$ $\Rightarrow (x-a)^2 = 4(a^2-y^2)$ $\Rightarrow x = a \pm 2\sqrt{a^2-y^2}$ $\Rightarrow x = a - 2\sqrt{a^2-y^2}\ (\because -a \le x \le a)$ $f^{-1}: x \to a - 2\sqrt{a^2-x^2}$, $x\in\mathbb{R}$, $0 \le x \le a$
wiv	$g(x) = f\left(\frac{3}{2}x\right)$ for $x\in\mathbb{R}$, $-\frac{2}{3}a \le x \le 2a$. Since $R_f = [0,a] \subseteq \left[-\frac{2}{3}a, 2a\right] = D_g$, the composite function gf exists.



$$D_{gf} = D_f \xrightarrow{f} R_f = [0, a] \xrightarrow{g} \left[\frac{\sqrt{3}}{2}a, a\right] = R_{gf}$$

$$R_{gf} = \left[\frac{\sqrt{3}}{2}a, a\right]$$

9i



$d(\frac{4}{5})^7 = 56.2$

$\therefore\ d = 267.9824829 = 268.0$ (1 d.p.)

ii

Since $|r| = \left|\frac{4}{5}\right| < 1$, sum to infinity exists.

Hence maximum theoretical length of wire

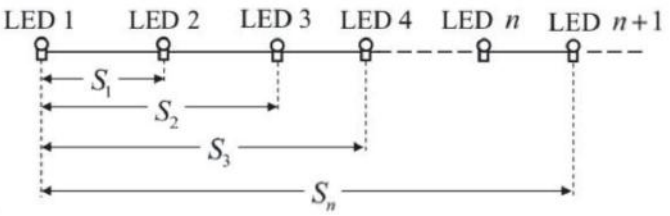
$= \frac{d}{1-r}$

$= \frac{267.9824829}{1-\frac{4}{5}}$

$= 1339.912415$

$= 1339.9$ cm (1 d.p.)

iii



Let S_n be distance from LED 1 to LED $n+1$.

Method 1: (using GC table)

Using sum of GP, $S_n = \dfrac{267.9824829[1-(\frac{4}{5})^n]}{1-\frac{4}{5}}$

Using GC,

n	S_n	$\lvert S_n - 1290\rvert$
14	1281.0	9.0
15	1292.8	2.8

$\therefore$ closest LED to 1290 is when $n = 15$.

$\therefore$ required LED is LED $(15+1)$ = LED 16.

Hence colour of LED 16 is red.

Method 2: (using algebraic manipulation)

Let $S_n = \dfrac{267.9824829[1-(\frac{4}{5})^n]}{1-\frac{4}{5}} = 1290$

$1-(\frac{4}{5})^n = 0.9627494943$

$n\ln(\frac{4}{5}) = \ln 0.0372505057$

$\therefore\ n = 14.74427443$

Using GC,

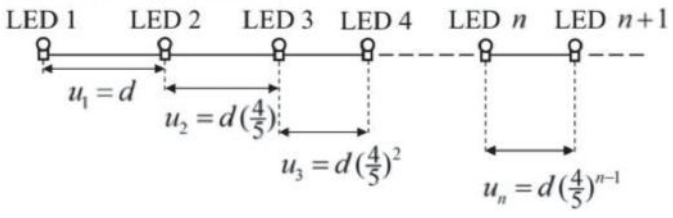
n	S_n	$\lvert S_n - 1290\rvert$
14	1281.0	9.0
15	1292.8	2.8

$\therefore$ closest LED to 1290 is when $n = 15$.

$\therefore$ required LED is LED $(15+1)$ = LED 16.

Hence colour of LED 16 is red.

iv



	Let $u_n = d\left(\frac{4}{5}\right)^{n-1} = 267.9824829\left(\frac{4}{5}\right)^{n-1} \geq 1$ $\left(\frac{4}{5}\right)^{n-1} \geq \frac{1}{267.9824829}$ $\ln\left(\frac{4}{5}\right)^{n-1} \geq \ln\frac{1}{267.9824829}$ $(n-1)\ln\left(\frac{4}{5}\right) \geq \ln\frac{1}{267.9824829}$ $n-1 \leq 25.05526859$ $\therefore \quad n \leq 26.05526859$ Since largest integer $n=26$, last LED is LED $(26+1)=$ LED 27 Hence colour of last LED 27 is yellow.
10a	$v+\mathrm{i}u=2$ ----(1) $av-2u=3\mathrm{i}$ ----(2) $(1)\times a-(2)$: $\mathrm{i}au+2u=2a-3\mathrm{i}$ $u=\frac{2a-3\mathrm{i}}{2+\mathrm{i}a}\times\frac{2-\mathrm{i}a}{2-\mathrm{i}a}$ $u=\frac{4a-6\mathrm{i}-2a^2\mathrm{i}-3a}{4+a^2}$ $u=\frac{a-(6+2a^2)\mathrm{i}}{4+a^2}$ $=\frac{a}{4+a^2}-\frac{6+2a^2}{4+a^2}\mathrm{i}$ Sub u into (1) $v=2-\mathrm{i}u$ $=2-\mathrm{i}\left(\frac{a}{4+a^2}-\frac{6+2a^2}{4+a^2}\mathrm{i}\right)$ $=2-\frac{6+2a^2}{4+a^2}-\frac{a}{4+a^2}\mathrm{i}$ $=\frac{2}{4+a^2}-\frac{a}{4+a^2}\mathrm{i}$

bi	Sub $x=-k$ into $bx^3+(12b+\mathrm{i})x^2+(12\mathrm{i}+b)x+12b=0$ We have: $-bk^3+(12b+\mathrm{i})k^2-(12\mathrm{i}+b)k+12b=0$ $-bk^3+12bk^2-bk+12b=0$ ---(1) and $k^2-12k=0$ ----(2) (2) implies $k(k-12)=0$ $k=0,12$ Reject $k=0$ as $b\neq 0$.	• • • •
bii	$bx^2+\mathrm{i}x+b=0$ $x=\dfrac{-\mathrm{i}\pm\sqrt{-1-4b^2}}{2b}$ $x=\dfrac{-1\pm\sqrt{1+4b^2}}{2b}\mathrm{i}$ Hence roots are purely imaginary	• • •
biii	Hence $bx^3+(12b+\mathrm{i})x^2+(12\mathrm{i}+b)x+12b=(x+12)(bx^2+cx+b)$ By compare coeff of x, $c=3\mathrm{i}$, using (i):	•

	$bx^3+(12b+\mathrm{i})x^2+(12\mathrm{i}+b)x+12b=(x+12)(bx^2+\mathrm{i}x+b)$ $=b(x+12)(x+\frac{1+\sqrt{1+4b^2}}{2b}\mathrm{i})(x+\frac{1-\sqrt{1+4b^2}}{2b}\mathrm{i})$	• •
11 a	Height of liquid in the cylinder $=x-4$ $V=\pi(2.5)^2(x-4)+\frac{1}{3}\pi(2.5)^2(4)$ $=6.25\pi(x-4)+\frac{25}{3}\pi$ $\frac{\mathrm{d}V}{\mathrm{d}x}=6.25\pi$ and $\frac{\mathrm{d}V}{\mathrm{d}t}=-18$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $=\frac{-18}{6.25\pi}$ $=-0.916$ cm/min	• •
b	$\frac{r}{x}=\frac{2.5}{4}\Rightarrow x=1.6r$ When $x=2$, $r=\frac{5}{4}$ $V_c=\frac{1}{3}\pi r^2 x$ $=\frac{1}{3}\pi r^2(1.6r)$ $=\frac{1.6}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}r}=1.6\pi r^2$	• •

	$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ $= \frac{1}{1.6\pi\left(\frac{5}{4}\right)^2} \times (-18)$ $= -2.29$ cm/min	
c	Vol of bowl $= \pi \int_{-10}^{0} 225\left(1 - \frac{y^2}{100}\right) dy$ Vol. of empty space $= \pi \int_{-10}^{0} 225\left(1 - \frac{y^2}{100}\right) dy - \pi(2.5)^2(20) - \frac{1}{3}\pi(2.5)^2(4)$ A B $= 4293.510$ cm^3 (3 d.p.)	• • •
12i	Equation of line through MD: $\mathbf{r} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ $\begin{pmatrix} 0.8 \\ 0.75 \\ 0.3 \end{pmatrix} \quad \begin{pmatrix} 0.4 \\ 0.9 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0.4 \\ 1.65 \\ 0.3 \end{pmatrix}$ Equation of line through ND: $\mathbf{r} = \begin{pmatrix} 0.4 \\ -0.9 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0.4 \\ 1.65 \\ 0.3 \end{pmatrix}$, $\lambda \in \mathbb{R}$. Since lines through MD and ND intersect at D,	St th lir **r** $\overline{M}$ W In (

$$\begin{cases} 0.8+2\lambda = 0.4+0.4\mu \\ 0.6+7\lambda = -0.9+1.65\mu \\ -\lambda = 0.3\mu \end{cases}$$

Solving using GC,

$\lambda = -\frac{3}{25}$, $\mu = \frac{2}{5}$.

Substitute $\lambda = -1.2$ into equation of line through MD,

$$\begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} - \frac{3}{25}\begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix}$$

$\therefore$ coordinates of D is $(0.56, -0.24, 0.12)$. (shown)

ii

Equation of horizontal ground: $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

Required angle

$$= 90° - \cos^{-1}\left|\frac{\begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{0.56^2 + (-0.24)^2 + 0.12^2}\sqrt{1}}\right|$$

$$= 11.14233567°$$

$$= 11.1° \quad (1 \text{ d.p.})$$

iii

Given $T(0, 0, 0.07)$.

$$\overrightarrow{NM} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.4 \\ -0.9 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 1.5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{TM} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0.07 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ -0.07 \end{pmatrix}$$

$$\begin{pmatrix} 0.4 \\ 1.5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0.8 \\ 0.6 \\ -0.07 \end{pmatrix} = \begin{pmatrix} -0.105 \\ 0.028 \\ -0.96 \end{pmatrix} = 0.01\begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix}$$

$\therefore$ equation of p:

$$\mathbf{r} \cdot \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = -6.72 \text{ (Shown)}$$

iv

Equation of line perpendicular to p passing through D is

$$\mathbf{r} = \begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix} + t \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix}, \ t \in \mathbb{R}.$$

Substitute equation of line into p,

$$\begin{pmatrix} 0.56 - 10.5t \\ -0.24 + 2.8t \\ 0.12 - 96t \end{pmatrix} \cdot \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = -6.72$$

$\therefore t = 0.00121618712$

Hence

$$\begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix} + 0.00121618712 \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} = \begin{pmatrix} 0.5472300352 \\ -0.2365946761 \\ 0.00324603648 \end{pmatrix}$$

$$= \begin{pmatrix} 0.547 \\ -0.237 \\ 0.00325 \end{pmatrix}$$

$\therefore$ coordinates of G is $(0.547, -0.237, 0.00325)$.

v

Method 1: [Hence using $(0.547, -0.237, 0.00325)$]

Required distance DG

$$= \sqrt{(0.56 - 0.54723)^2 + (-0.24 - (-0.23659))^2 + (0.12 - 0.003246)^2}$$

$$= 0.1174998$$

$$= 0.117 \quad (3 \text{ s.f.})$$

$\therefore$ required distance $= 117$ m

Method 2: [Otherwise using dot product]

$$\overrightarrow{MD} = \begin{pmatrix} 0.56 \\ -0.24 \\ 0.12 \end{pmatrix} - \begin{pmatrix} 0.8 \\ 0.6 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.24 \\ -0.84 \\ 0.12 \end{pmatrix}$$

Required distance

$$= \left| \begin{pmatrix} -0.24 \\ -0.84 \\ 0.12 \end{pmatrix} \cdot \frac{\begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix}}{\sqrt{(-10.5)^2 + 2.8^2 + (-96)^2}} \right|$$

$$= 0.1174996006$$

$$= 0.117 \quad (3 \text{ s.f.})$$

$\therefore$ required distance $= 117$ m

Method 3

$$\left|\overrightarrow{DG}\right| = \beta \left| \begin{pmatrix} -10.5 \\ 2.8 \\ -96 \end{pmatrix} \right|$$

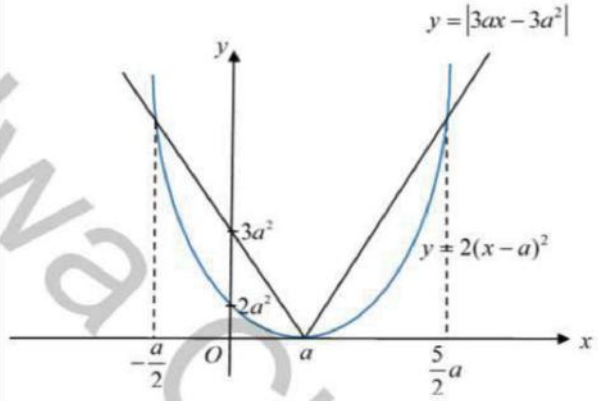
$= 0.00121618712\sqrt{9334.09}$

$= 0.1174996006$

$= 0.117$ (3 s.f.)

$\therefore$ required distance $= 117$ m

2019 C2 H2 Prelim P2 Solutions

Qn	Solutions	Commen
1		In genera part. Common 1. Fo in 2. Th stu of 3. So nu gr
1i	To find points of intersection, we solve $y=\lvert 3ax-3a^2\rvert$ $y=2(x-a)^2$ i.e. $2(x-a)^2=\lvert 3ax-3a^2\rvert$ $2(x-a)^2=3ax-3a^2$ -----(1) $\Rightarrow 2x^2-7ax+5a^2=0$ $\Rightarrow (2x-5a)(x-a)=0$ $\Rightarrow x=\frac{5}{2}a$ or $x=a$ Or $2(x-a)^2=-(3ax-3a^2)$ ------(2) $\Rightarrow 2x^2-ax-a^2=0$ $\Rightarrow (2x+a)(x-a)=0$ $\Rightarrow x=-\frac{1}{2}a$ or $x=a$ For $2(x-a)^2\geq\lvert 3ax-3a^2\rvert$, $x\leq -\frac{a}{2}$ or $x=a$ or $x\geq\frac{5a}{2}$	A lot of s previous Majority that $x=a$ The easie intercepts based on final conc A lot of s method to directly w manipula Common 1. $\lvert a$ m M lo 2. So w in fa Th (

1ii	Replace x with $x+\frac{a}{2}$, $2(x+\frac{a}{2}-a)^2 \geq \left\lvert 3a\left(x+\frac{a}{2}\right)-3a^2\right\rvert$ $2\left(x-\frac{a}{2}\right)^2 \geq \left\lvert 3ax-\frac{3a^2}{2}\right\rvert$ $\therefore\ x+\frac{a}{2} \leq -\frac{a}{2}$ or $x+\frac{a}{2}=a$ or $x+\frac{a}{2} \geq \frac{5a}{2}$ $\Rightarrow x \leq -a$ or $x=\frac{a}{2}$ or $x \geq 2a$	A lot replac
2i	$y=\frac{e^{\sin x}}{\sqrt{1+2x}}$ $y\sqrt{1+2x}=e^{\sin x}$ $\sqrt{1+2x}\frac{dy}{dx}+\frac{1}{2}\frac{2}{\sqrt{1+2x}}y=\cos x\, e^{\sin x}$ $\frac{dy}{dx}+\frac{y}{1+2x}=\frac{e^{\sin x}}{\sqrt{1+2x}}\cos x$ $=y\cos x$ $\frac{1}{y}\frac{dy}{dx}+\frac{1}{1+2x}=\cos x$ Alternatively, $\ln\left[y\sqrt{1+2x}\right]=\sin x$ $\frac{\sqrt{1+2x}\left(\frac{dy}{dx}\right)+(1+2x)^{-\frac{1}{2}}}{y\sqrt{1+2x}}=\cos x$ $\frac{1}{y}\frac{dy}{dx}+\frac{1}{1+2x}=\cos x$ (Shown)	This Stude detail quest credit

ii

Differentiating again w.r.t. x,

$$\frac{1}{y}\frac{d^2y}{dx^2}-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2-\frac{2}{(1+2x)^2}=-\sin x$$

$$\frac{1}{y}\frac{d^3y}{dx^3}-\frac{1}{y^2}\frac{dy}{dx}\frac{d^2y}{dx^2}-\frac{2}{y^2}\frac{dy}{dx}\frac{d^2y}{dx^2}+\frac{2}{y^3}\left(\frac{dy}{dx}\right)^3+\frac{8}{(1+2x)^3}=-\cos x$$

$$\frac{1}{y}\frac{d^3y}{dx^3}-\frac{3}{y^2}\frac{dy}{dx}\frac{d^2y}{dx^2}+\frac{2}{y^3}\left(\frac{dy}{dx}\right)^3+\frac{8}{(1+2x)^3}=-\cos x$$

When $x=0, y=1, \frac{dy}{dx}=0, \frac{d^2y}{dx^2}=2, \frac{d^3y}{dx^3}=-9$.

The Maclaurin's series for y is

$$y=1+\frac{2}{2!}x^2-\frac{9}{3!}x^3+\ldots$$

$$=1+x^2-\frac{3}{2}x^3+\ldots$$

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iii

$$\int_0^1 \frac{e^{\sin x}}{\sqrt{1+2x}}dx \approx \int_0^1 1+x^2-\frac{3}{2}x^3\,dx$$

$$=\frac{23}{24}$$

$$=0.985 \text{ (3 s.f.)}$$

A lot
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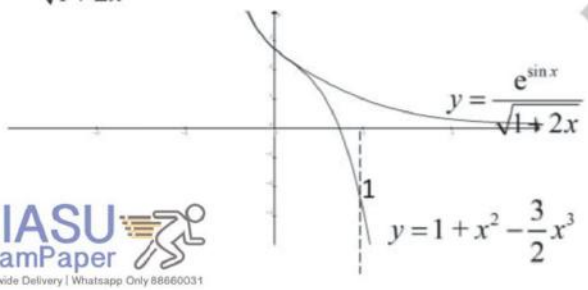
iii

$$\int_0^1 \frac{e^{\sin x}}{\sqrt{1+2x}}dx = 0.058$$



The graphs of $y=\frac{e^{\sin x}}{\sqrt{1+2x}}$ and $y=1+x^2-\frac{3}{2}x^3$ differ significantly near to $x=1$. Hence the approximation is not good.

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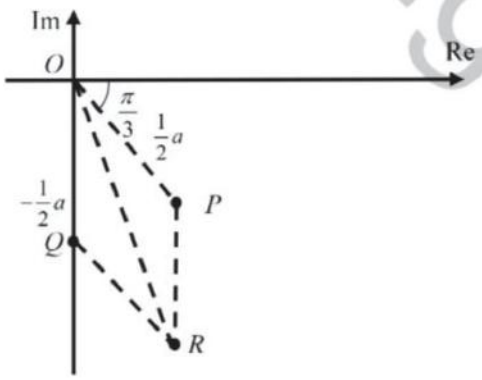
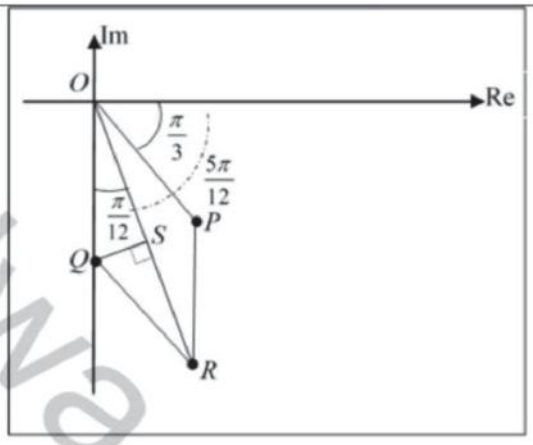
iv	Replace x with $-x$ $y=\dfrac{e^{\sin(-x)}}{\sqrt{1-2x}}=\dfrac{e^{-\sin x}}{\sqrt{1-2x}}=\dfrac{1}{e^{\sin x}\sqrt{1-2x}}$ $\dfrac{1}{e^{\sin x}\sqrt{1-2x}}\approx 1+x^2+\dfrac{3}{2}x^3$	Many series conne (ii).
3i	$\lvert p\rvert=\left\lvert\dfrac{a}{1+\sqrt{3}\mathrm{i}}\right\rvert=\dfrac{a}{2}$ $\arg(p)=\arg\left(\dfrac{a}{1+\sqrt{3}\mathrm{i}}\right)$ $=\arg(a)-\arg\left(1+\sqrt{3}\mathrm{i}\right)$ $=0-\dfrac{\pi}{3}$ $=-\dfrac{\pi}{3}$	• In ar m p. mi • A 4th • A
ii	 The shape is a rhombus.	• Al fo ve p sh • Sc di • A qu tri
ii	$\angle OQR=\dfrac{\pi}{2}-\dfrac{\pi}{3}=\dfrac{\pi}{12}$ $\arg(p+q)=-(\dfrac{\pi}{3}+\dfrac{\pi}{12})=-\dfrac{5\pi}{12}$	• Sc ar or ro • Sc ar an



$|p+q| = 2 \times OS = 2 \times \frac{a}{2} \cos\frac{\pi}{12} = a\cos\frac{\pi}{12}$

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- Sc
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iii

For $(p+q)^n$ to be purely imaginary,

$(p+q)^n = (a\cos\frac{\pi}{12})^n[\cos(-\frac{5n\pi}{12}) + i\sin(-\frac{5n\pi}{12})]$

$\cos(-\frac{5n\pi}{12}) = 0$

Hence the smallest positive integer n is 6, as $\cos(-\frac{5\pi}{2}) = 0$.

- Sc
  su
- a f

4ai

$\frac{dP}{dt} = \frac{a}{P} - bP$, where a, b are constants

When $P = 2$,

$\frac{dP}{dt} = \frac{a}{2} - 2b = 0 \Rightarrow a = 4b$

$\frac{dP}{dt} = \frac{4b}{P} - bP$

$= b\left(\frac{4}{P} - P\right)$

$= k\left(\frac{4}{P} - P\right)$, where $k = b$

4aii

When $P = 4$, $\frac{dP}{dt} = -3$.

$\frac{dP}{dt} = k\left(\frac{4}{4} - 4\right) = -3$

$\Rightarrow k = 1$

$\frac{dP}{dt} = \frac{4}{P} - P$

$= \frac{4 - P^2}{P}$

$\frac{dt}{dP} = \frac{P}{4 - P^2}$

$= -\frac{1}{2}\frac{-2P}{4 - P^2}$

$t = -\frac{1}{2}\ln|4 - P^2| + C$

$-2t = \ln|4 - P^2| - 2C$

$\ln|4 - P^2| = 2C - 2t$

$4 - P^2 = e^{2C}e^{-2t}$

$4 - P^2 = Ae^{-2t}$, where $A = \pm e^{2C}$

When $t = 0$, $P = 4$.

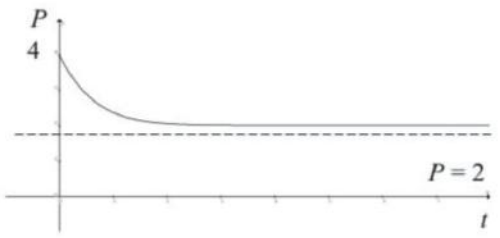
$A = -12$

$4 - P^2 = -12e^{-2t}$

$P^2 = 4 + 12e^{-2t}$

$P = 2\sqrt{1 + 3e^{-2t}}$

4a iii



The population of the bugs will decrease and approach 2000 in the long run.

4b

Method 1:

$$u = \frac{N}{t}$$

$$\frac{du}{dt} = \frac{t\frac{dN}{dt} - N}{t^2}$$

$$t\frac{du}{dt} = \frac{dN}{dt} - \frac{N}{t}$$

Substitute into $\frac{dN}{dt} = 4 + \frac{N}{t}$:

$$t\frac{du}{dt} = 4$$

$$\frac{du}{dt} = \frac{4}{t}$$

$$u = 4\ln t + C$$

$$\frac{N}{t} = 4\ln t + C$$

When $t = 1$, $N = 1 \Rightarrow C = 1$.

$N = 4t\ln t + t$

Method 2:

$$u = \frac{N}{t}$$

$$N = ut$$

$$\frac{dN}{dt} = \frac{du}{dt}t + u$$

Substitute into $\frac{dN}{dt} = 4 + \frac{N}{t}$:

$$\frac{du}{dt}t + u = 4 + u$$

$$\frac{du}{dt}t = 4$$

$$\frac{du}{dt} = \frac{4}{t}$$

$$u = 4\ln t + C$$

$$\frac{N}{t} = 4\ln t + C$$

When $t = 1$, $N = 1 \Rightarrow C = 1$.

$N = 4t\ln t + t$

5 (i)

Let X denote the daily rainfall in mm. $X \sim \mathrm{N}(\mu, \sigma^2)$

In 7 days, $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{7}\right)$

$\mathrm{P}(X > 9.8) = 0.1$

$$\mathrm{P}\left(Z > \frac{9.8-\mu}{\sigma}\right) = 0.1$$

$$\frac{9.8-\mu}{\sigma} = 1.2815516$$

$9.8 = \mu + 1.2815516\sigma$ --- (1)

$\mathrm{P}(\bar{X} > 8.2) = 0.1$

$$\mathrm{P}\left(Z > \frac{8.2-\mu}{\sigma/\sqrt{7}}\right) = 0.1$$

$$\frac{8.2-\mu}{\sigma/\sqrt{7}} = 1.2815516$$

$8.2 = \mu + 0.4843810\sigma$ --- (2)

Solving (1) and (2),

$\mu = 7.22780, \sigma = 2.00710$

$\therefore \sigma = 2.01\,(2\text{dp})$ (shown)

(ii)

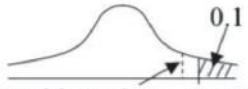
In 30 days, $\bar{X} \sim \mathrm{N}\left(7.22780, \frac{2.01^2}{30}\right)$

$\mathrm{P}(\bar{X} > k) \geq 0.1$

$k \leq 7.6981\,(5\text{sf})$

$\therefore \max k = 7.6\,(1\text{dp})$

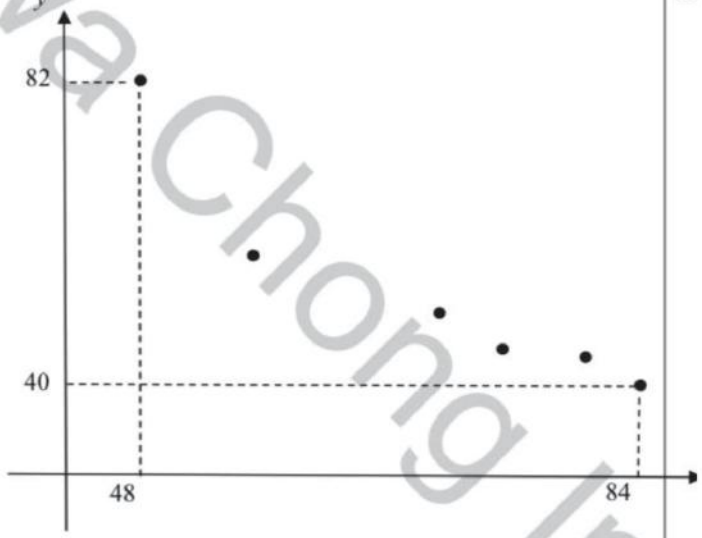
Note: change in inequality sign because for area to be greater than or equal to 0.1, the value of k can be at 7.6974 or to its left, i.e. less than 7.6974 (see diagram below).



Possible actual boundary 7.6974
to get area larger than 0.1

6

(i)



(ii)

(x_7, y_7) is $(\bar{x}, \bar{y})$

Using G.C,

$(\bar{x}, \bar{y}) = (69, 53)$

OR

$(x_7, y_7) = \left(\frac{80+84+70+74+58+48}{6}, \frac{44+40+49+45+58+82}{6}\right)$

$= (69, 53)$

(iii)

Model A: $y = a + bx^2$ r-value $= -0.92386$

Model B: $y = \ln a + b \ln x$ r-value $= -0.96785$

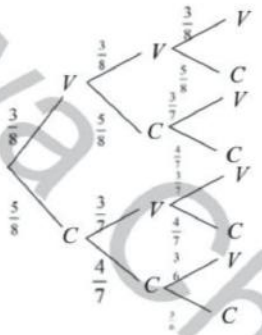
Model C: $y = a + b\sqrt{x}$ r-value $= -0.95855$

Since $|r|$ for Model B is nearest to 1, Model B is the most accurate model

(iv)	$y = \ln a + b \ln x$ By GC, $\ln a = 350.11$ $b = -70.469$ $y = 350.11 - 70.469 \ln x$ When $x = 60$, $y = 62$ marks. The student is estimated to score 62 marks.	N us of m A w A ec
(v)	$4 \times 30 = 120$ unreasonable as 120 is not within data range where $48 \le x \le 84$.	Sc he Th th ar th m
7 **(i)**	Let X and Y denote the length (in inches) of a "footlong" and a "6-inch" loaf respectively. $X \sim \mathrm{N}(12.2, 0.2^2)$, $Y \sim \mathrm{N}\left(6.1, \left(0.1\sqrt{2}\right)^2\right)$ $\mathrm{P}(Y < 6) = 0.239750 = 0.240(3\text{sf})$ $\mathrm{P}(X < 12) = 0.158655 = 0.159(3\text{sf}) < \mathrm{P}(Y < 6)$ $\therefore$ a "6-inch" loaf more likely to be less than 6 inches.	A cc va th 6, ga It pr St fu as qu
(ii)	$\mathrm{E}(Y_1 + Y_2 - X) = 2(6.1) - 12.2 = 0$ $\mathrm{Var}(Y_1 + Y_2 - X) = 2\left(0.1\sqrt{2}\right)^2 + 0.2^2 = 0.08$ $Y_1 + Y_2 - X \sim \mathrm{N}(0, 0.08)$ $\mathrm{P}(Y_1 + Y_2 > X) = \mathrm{P}(Y_1 + Y_2 - X > 0) = \frac{1}{2}$	It in M th th Sc w as ar

(iii)	Let A denote the number of "6-inch" sandwiches less than 6-inches in length in a week. $A \sim \mathrm{B}(3, 0.239750)$ $\mathrm{P}(A \le 1) = 0.855122 = 0.855(3\text{sf})$ Alternative $\text{required probability} = \mathrm{P}(Y < 6)\left[\mathrm{P}(Y > 6)\right]^2 \times 3 + \left[\mathrm{P}(Y > 6)\right]^3$ $= 0.23975(1-0.23975)^2 \times 3 + (1-0.23975)^3$ $= 0.855(3\text{sf})$	St va w or di of sa "1 Sc w m "x sa in
(iv)	Let B and C denote the number of "6-inch" sandwiches less than 6-inches in length in a 4-week and 3-week period respectively. $B \sim \mathrm{B}(12, 0.239750)$ and $C \sim \mathrm{B}(9, 0.239750)$ $\mathrm{P}(A = 1 \mid B > 4) = \dfrac{\mathrm{P}(A=1)\mathrm{P}(C>3)}{\mathrm{P}(B>4)}$ $= \dfrac{\mathrm{P}(A=1)\left[1-\mathrm{P}(C \le 3)\right]}{1-\mathrm{P}(B \le 4)}$ $= \dfrac{0.41571 \times 0.14710}{0.13721}$ $= 0.445674$ $= 0.446(3\text{sf})$	Th ch fr Sc co Sc de nu 4t P = Sc 2 up P — =
8 (a)(i)	Number of arrangements $= \dfrac{8!}{3!2!} = 3360$	Sc re
8 (a)(ii)	Number of arrangements $= \dfrac{5!}{2!} \times {}^{6}C_{3} = 1200$	M co 2
8 (a)(iii)	Arrange 3 vowels (into a group) $\dfrac{3!}{2!} = 3$	M es er

	Arrange 5 consonants (for slotting) $\frac{5!}{3!}=20$ Since the ends must be consonants, only the middle 4 slots can be used for the group to slot. (ie. ${}^4C_1$) Number of arrangements$=3\times20\times4=240$ OR Case 1: First and Last 'L' $4!\times\frac{3!}{2!}=72$ Case 2: First and last 'L' and non-L $2\times2\times\frac{4!}{2!}\times\frac{3!}{2!}=144$ Case 3: First and last non- L $2\times\frac{4!}{3!}\times\frac{3!}{2!}=24$ Total= 240 ways	
(b)(i)	P(1V) + P(2V) + P(3V) $=\frac{{}^5C_3\,{}^3C_1+{}^5C_2\,{}^3C_2+{}^5C_1\,{}^3C_3}{{}^8C_4}$ $=\frac{65}{70}=\frac{13}{14}$ Probability = 1 – P(all consonants) $=1-\left(\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\right)=\frac{13}{14}$ OR 1 – P(no vowels) $=1-\frac{{}^5C_4}{{}^8C_4}$ $=1-\frac{5}{70}$ $=\frac{13}{14}$	G

	OR P(1 vowel) + P(2 vowels) + P(3 vowels) $=\frac{3\times5\times4\times3}{8\times7\times6\times5}\times\frac{4!}{3!}+\frac{3\times2\times5\times4}{8\times7\times6\times5}\times\frac{4!}{2!2!}+\frac{3\times2\times1\times5}{8\times7\times6\times5}\times\frac{4!}{3!}$ $=\frac{13}{14}$	
8 **(b)(ii)**	 Probability = P(VCC)+P(CVC)+P(CCV)+P(CCC) $=\frac{3}{8}\times\frac{5}{8}\times\frac{4}{7}+\frac{5}{8}\times\frac{3}{7}\times\frac{4}{7}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}+\frac{5}{8}\times\frac{4}{7}\times\frac{3}{6}$ $=\frac{505}{784}$ $=0.644$	M di
9(i)	Using GC, Unbiased estimate of population mean is $\bar{x}=949.84$ (2 d.p.) Unbiased estimate of population variance is $s^2=1.31634^2=1.73$ (2 d.p.) Note: If student write $\mu=949.84$ or $\sigma^2=1.31634^2=1.73$, annotate but do not deduct.	G St au ar gi T m st 1. g 2.

9 **(a) (ii)**	Let X be the volume of shower gel dispensed by the machine. Let μ denote the population mean volume of shower gel dispensed by the machine. $\bar{X} \sim \mathrm{N}\left(\mu, \frac{s^2}{n}\right)$ approx. by Central Limit Theorem. $\bar{X} \sim \mathrm{N}\left(950, \frac{s^2}{n}\right)$ $H_0: \mu = 950$ $H_1: \mu > 950$	
9 **(a)(iii)**	Under H_0, using GC, $p - \text{value} = 0.13474 \approx 0.135$	

	The p - value is the probability of the sample mean volume of shower gel in a bottle is as extreme as 949.84 when the mean is actually 950.	T w 9 m M th cc S th w of hy st p- cc
9 (a)(iv)	Since we reject H_0 , $p-\text{value} < \alpha/100$ $\alpha \geq 13.476$ $\{\alpha \in \mathbb{R} : 13.5 \leq \alpha \leq 100\}$	B di ne up St us in ro th
9(b)	Let Y be the volume of shower gel dispensed by the machine after recalibration. Let μ denote the population mean volume of refill dispensed by the machine. $\bar{Y} \sim \text{N}\left(\mu, \frac{s^2}{n}\right)$ approx. by Central Limit Theorem H_0: $\mu = 250$ H_1: $\mu \neq 250$ Test statistic: $t = \frac{\bar{Y}-\mu}{s/\sqrt{n}}$ Level of significance: 1% Reject H_0 if $p-\text{value} \leq 0.01$	C th 1. 2. to 3. as cc 4.

	$\frac{249.5-250}{s/\sqrt{50}} \le -2.5758293$ or $\frac{249.5-250}{s/\sqrt{50}} \ge 2.5758293$ (NA) $0 < \frac{s}{\sqrt{50}} \le 0.19411$ $0 < s \le 1.3726$ $0 < \frac{1}{49}\left(k - \frac{(-25)^2}{50}\right) \le 1.3726^2$ $12.5 < k \le 104.8$	M lo w in M th 2 re th
9(c)	There is no need for the floor supervisor to assume the volume of shower gel follow a normal distribution as the sample sizes in both part (a) and (b) are large. The sample mean volume of shower gel can be approximated to follow a normal distribution by Central Limit Theorem.	A Tl sa ca nc Tl w 's St ar fo
10 **(i)**	Let $\$W$ denote the amount won by a player in one game. Under option A, $E(W_A) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right)$ $= (1+2+3+4)\left(\frac{1}{4}\right)$ $= \frac{5}{2} = \$2.50$ Under option B, the winning amount for each combination is listed below:	St ra di nc ar pr sc G ex *A* H nu kn w re as pc di cc

y / x	1	2	3	4
1	0	4	6	8
2	-1	0	12	16
3	-2	-1	0	24
4	-3	-2	-1	0

	$\mathrm{E}(W_B)=\left(\frac{1}{16}\right)[4+6+12+8+16+24$ $-1-2-3-1-2-1]$ $=\left(\frac{1}{16}\right)[70-10]$ $=\frac{15}{4}=\$3.75$ Since $\mathrm{E}(W_B)>\mathrm{E}(W_A)$, option B is better. (shown)	ca ar It ex ar O
(ii)	Option A is a "sure win" option where the player would definitely gain a positive amount in all cases, whereas option B has a risk of losing money in some cases.	G
(iii)	$\mathrm{E}(W_A^2)=(1^2+2^2+3^2+4^2)\left(\frac{1}{4}\right)=\frac{15}{2}$ $\mathrm{Var}(W_A)=\mathrm{E}(W_A^2)-[\mathrm{E}(W_A)]^2=\frac{15}{2}-\left(\frac{5}{2}\right)^2=\frac{5}{4}=1.25(\text{shown})$	G T st cc
(iv)	Let A and B denote the total amount won by Abel and Benson respectively. $A=W_{A1}+W_{A2}+\cdots+W_{A50}$ Since $n=50$ is large, by Central Limit Theorem, $A\sim\mathrm{N}\left(50\left(\frac{5}{2}\right),50\left(\frac{5}{4}\right)\right)$ approximately i.e. $A\sim\mathrm{N}\left(125,\frac{125}{2}\right)$ $B=W_{B1}+W_{B2}+\cdots+W_{B50}$ $B\sim\mathrm{N}\left(50\left(\frac{15}{4}\right),50\left(\frac{887}{16}\right)\right)$ approximately by Central Limit Theorem since $n=50$ is large $B\sim\mathrm{N}\left(\frac{375}{2},\frac{22175}{8}\right)$	A w va ar w th T ap of ar O - - - - -

		-
(v)	$A-B \sim \mathrm{N}\left(125-\frac{375}{2}, \frac{125}{2}+\frac{22175}{8}\right)$ $A-B \sim \mathrm{N}\left(-\frac{125}{2}, \frac{22675}{8}\right)$ $\mathrm{P}(A>B)=\mathrm{P}(A-B>0)$ $=0.120207$ $=0.120(3\mathrm{sf})$	M at re pr A cc ar fo - -