1 Electricity cost per household is calculated by multiplying the electricity consumption (in kWh), by the tariff (in cents/kWh). The tariff is set by the government and reviewed every 4 months.

The amount of electricity used by each household for each 4-month period, together with the total electricity cost for each household in the year, are given in the following table.

	Jan – April (in kWh)	May – Aug (in kWh)	Sept – Dec (in kWh)	Total electricity cost in the year ($)
Household 1	677	586	699	529.53
Household 2	1011	871	1048	790.63
Household 3	1349	1174	1417	1063.28

Write down and solve equations to find the tariff, in cents/kWh, to 2 decimal places, for each 4-month period. [4]

2 A string of fixed length l is cut into two pieces. The first piece is used to form a square of side s and the second piece is used to form a circle of radius r. Find the ratio of the length of the first piece to the second piece that gives the smallest possible combined area of the square and circle. [6]

3 A geometric progression has first term a and common ratio r, and an arithmetic progression has first term a and common difference d, where a and d are non-zero. The sums of the first 2 and 4 terms of the arithmetic progression are equal to the respective sums of the first 2 and 4 terms of the geometric progression.

(i) By showing $r^3+r^2-5r+3=0$, or otherwise, find the value of the common ratio. [5]

(ii) Given that $a<0$ and the nth term of the geometric progression is positive, find the smallest possible value of n such that the nth term of the geometric progression is more than 1000 times the nth term of the arithmetic progression. [3]

4 **(i)** Find the series expansion for $(1+ax)^n$ in ascending powers of x, up to and including the term in x^3, where a is non-zero and $|a|<1$. [1]

(ii) It is given that the coefficients of the terms in x, x^2, and x^3 are three consecutive terms in a geometric progression. Show that $n=-1$. [2]

(iii) Show that the coefficients of the terms in the series expansion of $(1+ax)^{-1}$ form a geometric progression. [3]

(iv) Evaluate the sum to infinity of the coefficients of the terms in x of odd powers. [2]

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2019

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME

CLASS

INDEX NO.

MATHEMATICS 9758/01

04 September 2019

Paper 1 [100 marks]

3 hours

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **27** printed pages (including this cover page) and **1** blank page.

For markers' use:											
Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total

5 **(a)** It is given that the equation $\mathrm{f}(x)=a$ has three roots x_1, x_2, x_3 where $x_1<0<x_2<x_3$, and a is a constant.

(i) How many roots does the equation $\mathrm{f}(|x|)=a$ have? With the aid of a diagram, or otherwise, explain your answer briefly. [2]

(ii) How many roots does the equation $\mathrm{f}(x-a)=a$ have? With the aid of a diagram, or otherwise, explain your answer briefly. [2]

(b) Solve the inequality $\frac{2\ln 2}{3\pi}x \leq |\ln(1-\sin x)|$, where $0 \leq x < 2\pi$. [4]

6 The curve C has equation $y=\frac{x^2+5x+3}{x+1}$.

(i) Show algebraically that the curve C has no stationary points. [2]

(ii) Sketch the curve C, indicating the equations of any asymptotes, and the coordinates of points where C intersects the axes. [4]

(iii) Region S is bounded by C, the y-axis, and the line $y=\frac{9}{2}$. Find the volume of the solid formed when region S is rotated about the x-axis completely. [3]

7 In the Argand diagram, the points P_1 and P_2 represent the complex numbers z and z^2 respectively, where $z=\sqrt{3}+\mathrm{i}\sqrt{3}$.

(i) Find the exact modulus and argument of z. [2]

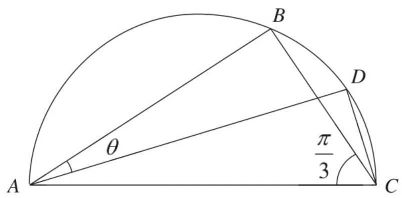
(ii) Mark the points P_1 and P_2 on an Argand diagram and find the area of the triangle OP_1P_2, where O represents the complex number 0. [3]

Let $w=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$.

(iii) Find the set of integer values n such that $\arg\left(w^n z^3\right)=-\frac{\pi}{4}$. [4]

8 **(a)** Given that $2^y = 2 + \sin 2x$, use repeated differentiation to find the Maclaurin series for y, up to and including the term in x^2. [5]

(b)



The points *A*, *B*, *C*, and *D* lie on a semi-circle with *AC* as its diameter. Furthermore, angle $DAB = \theta$, and angle $ACB = \dfrac{\pi}{3}$.

(i) Show that $\dfrac{BC}{DC} = \dfrac{1}{\cos\theta - \sqrt{3}\sin\theta}$. [3]

(ii) Given that θ is a sufficiently small angle, show that

$$\frac{BC}{DC} \approx 1 + a\theta + b\theta^2,$$

for constants a and b to be determined. [3]

9 **(a)** **(i)** Find $\int 2\sin x\cos 3x \, dx$. [3]

(ii) Hence, show that $\int 2x\sin x\cos 3x \, dx = \frac{1}{16}\left[-4x\cos 4x + 8x\cos 2x + \sin 4x - 4\sin 2x\right] + C$, where C is an arbitrary constant. [3]

(b) The curve *C* has parametric equations

$$x = \theta^2,\ y = \sin\theta\cos 3\theta,\ \text{where } 0 \le \theta \le \frac{\pi}{2}.$$

(i) Sketch the curve *C*, giving the exact coordinates of the points where it intersects the *x*-axis. [2]

(ii) By using the result in **(a)(ii)**, find the exact total area of the regions bounded by the curve *C* and the *x*-axis. [4]

10 An object is heated up by placing it on a hotplate kept at a high temperature. A simple model for the temperature of the object over time is given by the differential equation

$$\frac{dT}{dt} = k(T_H - T),$$

where T is the temperature of the object in degrees Celsius, T_H is the temperature of the hotplate in degrees Celsius, t is time measured in seconds and k is a real constant.

(i) State the sign of k and explain your answer. [1]

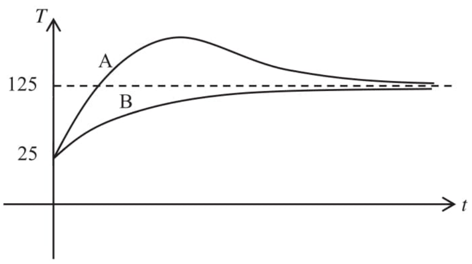
(ii) It is given that the temperature of the object is 25 degrees Celsius at $t = 0$, and the temperature of the hotplate is kept constant at 275 degrees Celsius. If the temperature of the object is 75 degrees Celsius at $t = 100$, find T in terms of t, giving the value of k to 5 significant figures. [6]

The model is now modified to account for heat lost by the object to its surroundings. The new model is given by the equation

$$\frac{dT}{dt} = k(T_H - T) - m(T - T_S),$$

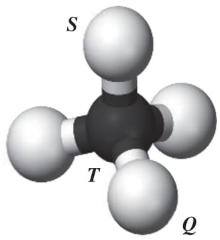
where T_S is the temperature of the surrounding environment in degrees Celsius and m is a positive real constant.

(iii) It is given that the object eventually approaches an equilibrium temperature of 125 degrees Celsius, and that the surrounding environment has a constant temperature which is lower than 125 degrees Celsius. One of the two curves (A and B) shown below is a possible graph of the object's temperature over time. State which curve this is, and explain clearly why the other curve cannot be a graph of the object's temperature over time. [2]



(iv) Using the same value of k as found in part **(ii)** and assuming $T_S = 25$, find the value of m. (You need not solve the revised differential equation.) [3]

11



Methane (CH_4) is an example of a chemical compound with a tetrahedral structure. The 4 hydrogen (H) atoms form a regular tetrahedron, and the carbon (C) atom is in the centre.

Let the 4 H-atoms be at points *P*, *Q*, *R*, and *S* with coordinates $(9,2,9)$, $(9,8,3)$, $(3,2,3)$, and $(3,8,9)$ respectively.

(i) Find a Cartesian equation of the plane Π_1 which contains the points *P*, *Q* and *R*. [4]

(ii) Find a Cartesian equation of the plane Π_2 which passes through the midpoint of *PQ* and is perpendicular to $\overrightarrow{PQ}$. [2]

(iii) Find the coordinates of point *F*, the foot of the perpendicular from *S* to Π_1. [4]

(iv) Let *T* be the point representing the carbon (C) atom. Given that point *T* is equidistant from the points *P*, *Q*, *R* and *S*, find the coordinates of *T*. [3]

End of Paper

EUNOIA JUNIOR COLLEGE
JC2 Preliminary Examination 2019
General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CLASS INDEX NO.

MATHEMATICS 9758/02

18 September 2019

Paper 2 [100 marks] 3 hours

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.
Write in dark blue or black pen.
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The number of marks is given in brackets [ ] at the end of each question or part question.

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For markers' use:											
Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total

Section A: Pure Mathematics [40 marks]

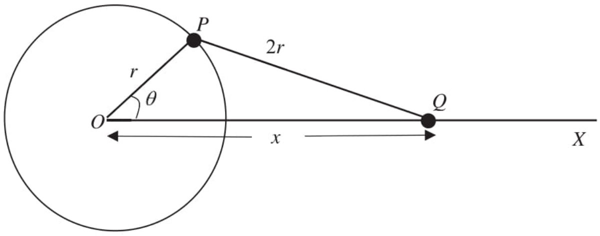
1 **(a)** Find the complex numbers z and w that satisfy the equations

$$\frac{z}{w} = 2 + 2\mathrm{i},$$

$$(1 - 2\mathrm{i})z = 39 - (11\mathrm{i})w.$$ [3]

(b) It is given that $(1 + \mathrm{i}c)^3$ is real, where c is also real. By first expressing $(1 + \mathrm{i}c)^3$ in Cartesian form, find all possible values of c. [3]

2



The diagram shows a mechanism for converting rotational motion into linear motion. The point P is on the circumference of a disc of fixed radius r which can rotate about a fixed point O. The point Q can only move on the line OX, and P and Q are connected by a rod of length $2r$. As the disc rotates, the point Q is made to slide along OX. At time t, angle POQ is θ and the distance OQ is x.

(i) State the maximum and minimum values of x. [1]

(ii) Show that $x = r\left(\cos\theta + \sqrt{4 - \sin^2\theta}\right)$. [2]

(iii) At a particular instant, $\theta = \dfrac{\pi}{6}$ and $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0.3$. Find the numerical rate at which point Q is moving towards point O at that instant, leaving your answer in terms of r. [3]

3 The position vectors of points P and Q, with respect to the origin O, are $\mathbf{p}$ and $\mathbf{q}$ respectively. Point R, with position vector $\mathbf{r}$, is on PQ produced, such that $3\overrightarrow{PR} = 5\overrightarrow{PQ}$.

(i) Given that $|\mathbf{p}| = \sqrt{29}$ and $\mathbf{p.r} = 11$, find the length of projection of $\overrightarrow{OQ}$ onto $\overrightarrow{OP}$. [4]

(ii) S is another point such that $\overrightarrow{PS} = \mathbf{r}$. Given that $\mathbf{p} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find the area of the quadrilateral $OPSR$. [3]

4 The ceiling function maps a real number x to the least integer greater than or equal to x. Denote the ceiling function as $\lceil x \rceil$. For example, $\lceil 2.1 \rceil = 3$ and $\lceil -3.8 \rceil = -3$.

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} \lceil x \rceil & \text{for } x \in \mathbb{R}, \quad -2 < x \le 1, \\ 0 & \text{for } x \in \mathbb{R}, \quad 1 < x \le 2. \end{cases}$$

(i) Find the value of $\mathrm{f}(-1.4)$. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 < x \le 2$. [2]

(iii) Does f^{-1} exist? Justify your answer. [1]

(iv) Find the range of f. [1]

The function g is defined as $\mathrm{g} : x \mapsto \dfrac{ax-3}{x-a}$, $x \in \mathbb{R}, x \ne a$, where $a > 0, a \ne 3$.

(v) Find $\mathrm{g}^2(x)$. Hence, or otherwise, evaluate $\mathrm{g}^{2019}(5)$, leaving your answer in a if necessary. [4]

(vi) Given that $a = 3$, find the range of gf. [1]

5 **(a)** The r^{th} term of a sequence is given by $u_r = \dfrac{4}{M^{3r-1}}$, where $M > 1$.

(i) Write down the first three terms of u_r in terms of M. [1]

(ii) Show that $\displaystyle\sum_{r=1}^{n} u_r = \frac{4M}{M^3 - 1}\left(1 - \frac{1}{M^{3n}}\right)$. [2]

(iii) Give a reason why the series in **(ii)** is convergent and state the sum to infinity. [2]

(b) **(i)** Show that $\cos\left(\dfrac{2r+1}{2}\right) - \cos\left(\dfrac{2r-1}{2}\right) = -2\sin\left(\dfrac{1}{2}\right)\sin(r)$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n} \sin r = \operatorname{cosec}\left(\frac{1}{2}\right)\sin\frac{n+1}{2}\sin\frac{n}{2}$. [4]

Section B: Probability and Statistics [60 marks]

6 A biased tetrahedral die has four faces, marked with the numbers 1, 2, 3 and 4. On any throw, the probability of the die landing on each face is shown in the table below, where c and d are real numbers.

Number on face	1	2	3	4
Probability of landing on face	0.3	c	d	0.2

(i) Write down an expression for d in terms of c. [1]

(ii) By writing the variance of the result of one throw of the die in the form $-\alpha\left(c-h\right)^2+k$, where α, h and k are positive constants to be determined, find the value of c which maximises this variance. [5]

(iii) If $c=0.2$, find the probability that in 10 throws of the die, at least 7 throws land on an even number. [2]

7 $X_1, X_2, X_3, \ldots$ are independent normally-distributed random variables with common mean μ and **different** variances. For each positive integer n, $\text{Var}(X_n)=2n$.

(i) Find $\text{P}(\mu-1<X_2<\mu+1)$. [3]

(ii) Find $\text{P}(X_3 \geq X_4)$. [1]

(iii) For each n, let $Y_n=\dfrac{X_1+X_2+\ldots+X_n}{n}$. By finding the distribution of Y_n in terms of n, determine the smallest integer value of n such that $\text{P}(\mu-1<Y_n<\mu+1)>\dfrac{2}{3}$. [4]

8 For events A and B, it is given that $\text{P}\left(A\right)=\dfrac{2}{5}$, $\text{P}\left(A\cup B\right)=\dfrac{6}{7}$, and $\text{P}\left(A\cap B'\right)=\dfrac{1}{3}$. Find:

(i) $\text{P}\left(B\right)$; [2]

(ii) $\text{P}\left(A' \mid B\right)$. [2]

A third event, C, is such that B and C are independent, and $\text{P}\left(C\right)=\dfrac{2}{5}$.

(iii) Find $\text{P}\left(B'\cap C\right)$. [2]

(iv) Hence, find the greatest and least possible values of $\text{P}\left(A\cap B'\cap C\right)$. [4]

9 **(a)** Find the number of ways of arranging the letters of the word JEWELLERY, if:

(i) there are no restrictions. [1]

(ii) the arrangement starts with 'L', and between any two 'E's there must be at least 2 other letters. [3]

A 4-letter 'codeword' is formed by taking an arrangement of 4 letters from the word JEWELLERY.

(iii) Find the number of 4-letter codewords that can be formed. [3]

(b) Mr and Mrs Lee, their three children, and 5 others are seated at a round table during a wedding dinner. Find the number of ways that everyone can be seated, such that Mr and Mrs Lee are seated together, but their children are not all seated together. [3]

10 A company manufactures packets of potato chips with X mg of sodium in each packet. It is known that the mean amount of sodium per packet is 1053 mg. After some alterations to the production workflow, 50 randomly chosen packets of potato chips were selected for analysis. The amount of sodium in each packet was measured, and the results are summarised as follows:

$$\sum(x-1050)=58.0,\ \sum(x-1050)^2=2326$$

(i) Test at 5% level of significance whether the mean amount of sodium in a packet of potato chips has changed, after the alterations to the workflow. [6]

(ii) Explain what '5% level of significance' means in this context. [1]

(iii) A second tester conducted the same test at $\alpha\%$ level of significance, for some integer α. However, he came to a different conclusion from the first tester. What is the range of α for which the second tester could have taken? [1]

(iv) Without performing another hypothesis test, explain whether the conclusion in part **(i)** would be different if the alternative hypothesis was that the mean amount of sodium had decreased after the alterations to the workflow. [1]

It is given instead that the standard deviation of amount of sodium in a packet of potato chips is 6.0 mg. The mean amount of sodium of a second randomly chosen sample of 40 packets of potato chips is $\bar{y}$.

(v) A hypothesis test on this sample, at 5% level of significance, led to a conclusion that the amount of sodium has decreased.

Find the range of values of $\bar{y}$, to 1 decimal place.

Explain if it is necessary to make any assumptions about the distribution of the amount of sodium in each packet of potato chips. [3]

11 **(a)** Comment on the following statement: "The product moment correlation coefficient between the amount of red wine intake and the risk of heart disease is approximately -1. Thus we can conclude that red wine intake decreases the risk of heart disease." [1]

(b) During an experiment, the radiation intensity, I, from a source at time t, in appropriate units, is measured and the results are tabulated below.

t	0.2	0.4	0.6	0.8	1.0
I	2.81	1.64	0.93	0.55	0.30

(i) Identify the independent variable and explain why it is independent. [1]

(ii) Draw a scatter diagram of these data. With the help of your diagram, explain whether the relationship between I and t is likely to be well modelled by an equation of the form $I = at + b$, where a and b are constants. [3]

(iii) Calculate, to 4 decimal places, the product moment correlation coefficient between

(a) I and t,

(b) $\ln I$ and t. [2]

(iv) Using the model $I = ae^{bt}$, find the equation of a suitable regression line, and calculate the values of a and b. [3]

(v) Use the regression line found in **(iv)** to estimate the radiation intensity when $t = 0.7$. Comment on the reliability of your estimate. [2]

End of Paper

EJC_H2_2019_JC2_Prelim_P1_Solutions

1	Let x, y, z be the tariff in ¢/kWh in Jan-Apr, May-Aug and Sept-Dec respectively. $677x+586y+699z=52953$ --- (1) $1011x+871y+1048z=79063$ --- (2) $1349x+1174y+1417z=106328$ --- (3) Solving with GC, $x=25.81$ $y=29.68$ $z=25.88$

2	Total length is l, thus we have $4s+2\pi r=l$ …(*) Let the combined area be A. $A=s^2+\pi r^2$ …(#) Method 1: implicit differentiation of (#) Use (#) to find $\frac{\mathrm{d}A}{\mathrm{d}s}$: $A=s^2+\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}s}=2s+2\pi r\frac{\mathrm{d}r}{\mathrm{d}s}$ Use (*) to find $\frac{\mathrm{d}r}{\mathrm{d}s}$: $4s+2\pi r=l \Rightarrow 4+2\pi\frac{\mathrm{d}r}{\mathrm{d}s}=0$ Thus $\frac{\mathrm{d}r}{\mathrm{d}s}=-\frac{2}{\pi}$ Sub into $\frac{\mathrm{d}A}{\mathrm{d}s}$: $\frac{\mathrm{d}A}{\mathrm{d}s}=2s+2\pi r\left(-\frac{2}{\pi}\right)=2s-4r$ For stationary value of A, $\frac{\mathrm{d}A}{\mathrm{d}s}=0 \Rightarrow s=2r$ Check minimum: $\frac{\mathrm{d}^2A}{\mathrm{d}s^2}=2-4\frac{\mathrm{d}r}{\mathrm{d}s}=2-4\left(-\frac{2}{\pi}\right)>0$ Thus A is a minimum when $s=2r$. Required ratio is $\frac{\text{length of first piece}}{\text{length of second piece}}=\frac{4s}{2\pi r}=\frac{4(2r)}{2\pi r}=\frac{4}{\pi}$ Method 2: differentiation in 1 variable 2a: expressing A in terms of r From (*), $s=\frac{l-2\pi r}{4}$ Sub into (#): $A=\pi r^2+\left(\frac{l-2\pi r}{4}\right)^2$

$$\frac{dA}{dr} = 2\pi r + 2\left(\frac{l-2\pi r}{4}\right)\left(\frac{-2\pi}{4}\right)$$

$$= \frac{\pi}{4}\left[8r - (l - 2\pi r)\right] = \frac{\pi}{4}\left[r(8+2\pi) - l\right]$$

For stationary value of A, $\frac{dA}{dr} = 0$:

$$\frac{\pi}{4}\left[r(8+2\pi) - l\right] = 0 \Rightarrow r = \frac{l}{8+2\pi}$$

Check minimum:

EITHER 2nd Derivative Test

$$\frac{d^2A}{dr^2} = \frac{\pi}{4}(8+2\pi) > 0$$

So $r = \frac{l}{8+2\pi}$ gives a minimum value of A.

OR 1st Derivative Test

$$\frac{dA}{dr} = \frac{\pi}{4}\left[r(8+2\pi) - l\right] = \frac{\pi}{4}(8+2\pi)\left(r - \frac{l}{8+2\pi}\right)$$

r	$\left(\frac{l}{8+2\pi}\right)^-$	$\left(\frac{l}{8+2\pi}\right)$	$\left(\frac{l}{8+2\pi}\right)^+$
$r - \frac{l}{8+2\pi}$	-ve	0	+ve
$\frac{dA}{dr}$	-ve	0	+ve

$$\frac{\text{length of first piece}}{\text{length of second piece}} = \frac{4s}{2\pi r} = \frac{l-2\pi r}{2\pi r} = \frac{l}{2\pi r} - 1$$

When A is minimum,

$$\frac{\text{length of first piece}}{\text{length of second piece}} = \frac{l}{2\pi\left(\frac{l}{8+2\pi}\right)} - 1$$

$$= \frac{(8+2\pi)}{2\pi} - 1$$

$$= \frac{4}{\pi}$$

<u>2b: expressing A in terms of s</u>

From (*), $r = \frac{l-4s}{2\pi}$

Sub into (#): $A = \pi\left(\frac{l-4s}{2\pi}\right)^2 + s^2$

$$\frac{dA}{dS} = 2\pi\left(\frac{l-4s}{2\pi}\right)\left(\frac{-4}{2\pi}\right) + 2s$$

$$= \frac{2}{\pi}\left[s(4+\pi) - l\right]$$

	For stationary value of A, $\frac{\mathrm{d}A}{\mathrm{d}s}=0$: $\frac{2}{\pi}\left[s(4+\pi)-l\right]=0 \Rightarrow s=\frac{l}{4+\pi}$ *Check minimum:* *EITHER 2nd Derivative Test* $\frac{\mathrm{d}^2A}{\mathrm{d}s^2}=\frac{2}{\pi}(4+\pi)>0$ So $s=\frac{l}{4+\pi}$ gives a minimum value of A OR 1st Derivative Test $\frac{\mathrm{d}A}{\mathrm{d}S}=\frac{2}{\pi}\left[s(4+\pi)-l\right]=\frac{2(4+\pi)}{\pi}\left(s-\frac{l}{4+\pi}\right)$

S	$\left(\frac{l}{4+\pi}\right)^-$	$\left(\frac{l}{4+\pi}\right)$	$\left(\frac{l}{4+\pi}\right)^+$
$s-\frac{l}{4+\pi}$	-ve	0	+ve
$\frac{\mathrm{d}A}{\mathrm{d}s}$	-ve	0	+ve

$$\frac{\text{length of first piece}}{\text{length of second piece}}=\frac{4s}{2\pi r}=\frac{4s}{2\pi\left(\frac{l-4s}{2\pi}\right)}=\frac{4s}{l-4s}$$

When A is minimum,

$$\frac{\text{length of first piece}}{\text{length of second piece}}=\frac{4\left(\frac{l}{4+\pi}\right)}{l-\frac{4l}{4+\pi}}$$
$$=\frac{4l}{4+\pi}\times\frac{4+\pi}{\pi l}$$
$$=\frac{4}{\pi}$$

Other possible methods:
Method 3: Differentiate w.r.t. ratio
Method 4: Complete the square

3	(i)

$\frac{2}{2}\left[2a+(2-1)d\right]=\frac{a\left(r^2-1\right)}{r-1}$

$\Rightarrow 2a+d=a(r+1)-------(1)$

$\frac{4}{2}\left[2a+(4-1)d\right]=\frac{a\left(r^4-1\right)}{r-1}$

$\Rightarrow 4a+6d=a\left(r^2+1\right)(r+1)-------(2)$

From (1): Sub $d=a(r+1)-2a$ into (2)

i.e. $d=ar-a$

$4a+6[ar-a]=a\left(r^2+1\right)(r+1)$

$6ar-2a=a\left(r^3+r^2+r+1\right)$

$r^3+r^2-5r+3=0$ (shown)

$(r-1)^2(r+3)=0$

$r=-3$ or $r=1$ (rej)

[If $r=1$, $d=a(r+1)-2a=0$, but $d\neq 0$]

<u>Alternative solution</u>

$2a+d=\frac{a\left(r^2-1\right)}{r-1}$ -- (1)

$4a+6d=\frac{a\left(r^4-1\right)}{r-1}$ -- (2)

6 x (1)-2:

$8a=\frac{6a\left(r^2-1\right)}{r-1}-\frac{a\left(r^4-1\right)}{r-1}$

$8a(r-1)=a\left(6r^2-6-r^4+1\right)$

$8r-8=6r^2-r^4-5$

$r^4-6r^2+8r-3=0$

Solving:

$r=-3$ or 1 (rej)

(ii)

$a(-3)^{n-1} > 1000[a+(n-1)d]$

Note: $d = a(-3+1)-2a = -4a$

$a(-3)^{n-1} > 1000[a+(n-1)(-4a)]$

$a(-3)^{n-1} > 1000a(5-4n)$

Since $a<0$,

$(-3)^{n-1} < 1000(5-4n)$

Since n^{th} term of GP is positive, i.e. $a(-3)^{n-1} > 0$

$(-3)^{n-1}$ is negative $\Rightarrow n$ is even

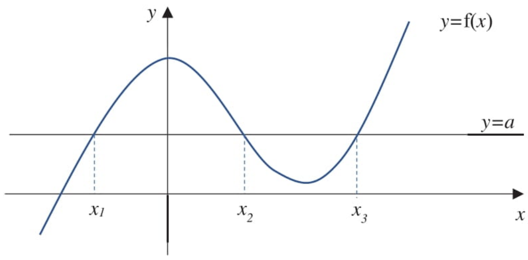
From GC (table)

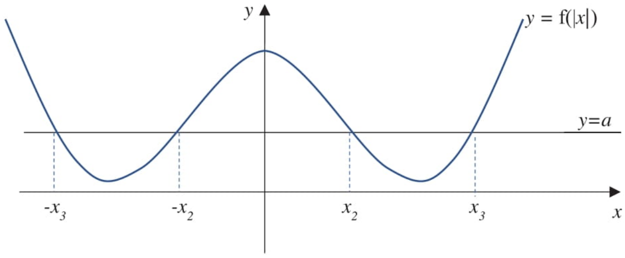
n	$(-3)^{n-1}$	$1000(5-4n)$
10	−19683 >	−35000
12	−177147 <	− 43000

Smallest $n=12$

4	(i) $(1+ax)^n = 1+n(ax)+\frac{n(n-1)}{2!}(ax)^2+\frac{n(n-1)(n-2)}{3!}(ax)^3+\ldots$
	(ii) Since the three coefficients form a GP, we have $\frac{\frac{n(n-1)}{2}a^2}{na} = \frac{\frac{n(n-1)(n-2)}{6}a^3}{\frac{n(n-1)}{2}a^2}$ $\frac{(n-1)a}{2} = \frac{(n-2)a}{3}$ $3(n-1) = 2(n-2)$ $n=-1$
	(iii) To prove GP, we need to show that $\frac{u_r}{u_{r-1}} = \text{constant}$ $u_r = \frac{n(n-1)\ldots(n-r+1)}{r!}a^r = \frac{(-1)(-2)\ldots(-1-r+1)}{r!}a^r$ $u_{r-1} = \frac{n(n-1)\ldots(n-r+2)}{(r-1)!}a^{r-1} = \frac{(-1)(-2)\ldots(-1-r+2)}{(r-1)!}a^{r-1}$ $\frac{u_r}{u_{r-1}} = \frac{(-1-r+1)}{r}a$ Since $n=-1$, $\frac{u_r}{u_{r-1}} = \frac{(-1-r+1)}{r}a = -a$ (constant) Alternatively, $(1+ax)^{-1} = 1-ax+a^2x^2-a^3x^3+\ldots+(-a)^{r-1}+\ldots$ $\frac{u_r}{u_{r-1}} = \frac{(-a)^{r-1}}{(-a)^{r-2}} = \frac{(-1)^{r-1}(a)^{r-1}}{(-1)^{r-2}(a)^{r-2}} = -a$ (constant)
	(iv) The cofficients of the terms in x of odd powers form a GP with first term $-a$, and common ratio a^2. Sum to infinity $= \frac{-a}{1-a^2} = \frac{a}{a^2-1}$

5 (a)(i)

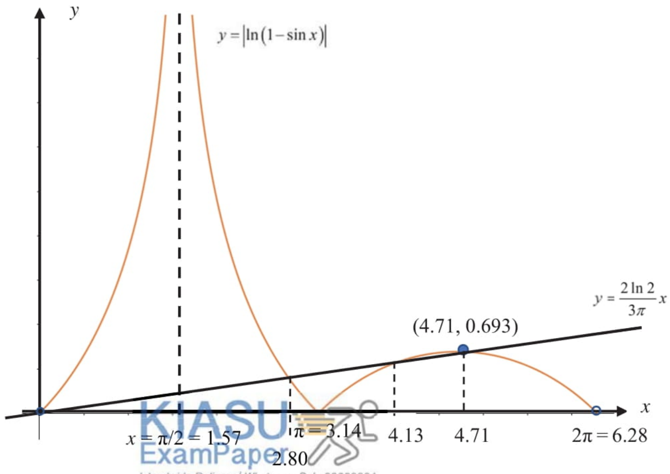




Since the graph of $y=\mathrm{f}(|x|)$ retains the part with positive x-values, $\mathrm{f}(|-x_2|)=\mathrm{f}(x_2)=a$. Similarly for $-x_3$.

Thus there will be 4 roots, i.e. $x_2, x_3, -x_2, -x_3$

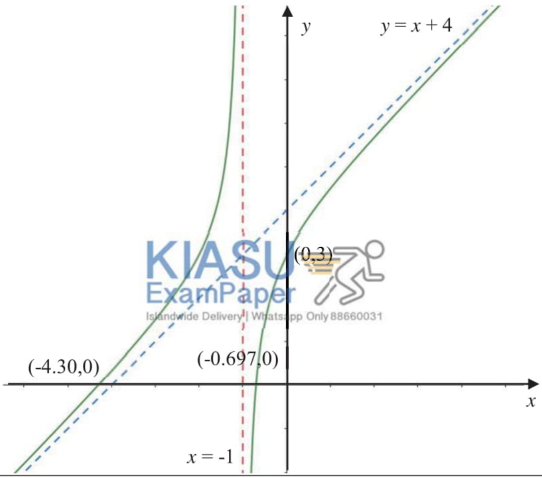
(b)



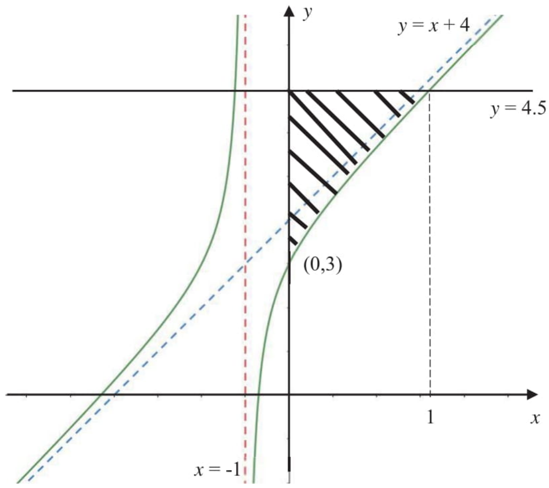
$$\frac{2\ln 2}{3\pi}x \le |\ln(1-\sin x)|$$

From the graph,

	$\frac{2\ln 2}{3\pi}x \le \left\|\ln(1-\sin x)\right\|$ $0 \le x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \le 2.80$ or $4.13 \le x \le \frac{3\pi}{2}$ OR $0 \le x < 1.57$ or $1.57 < x \le 2.80$ or $4.13 \le x \le 4.71$ (3 s.f)

6	(i) $y = \frac{x^2+5x+3}{x+1} = x+4-\frac{1}{x+1}$ $\frac{dy}{dx} = 1+(x+1)^{-2} = 1+\frac{1}{(x+1)^2}$ or $\frac{dy}{dx} = \frac{(x+1)(2x+5)-(x^2+5x+3)(1)}{(x+1)^2}$ $= \frac{x^2+2x+2}{(x+1)^2} = \frac{1+(x+1)^2}{(x+1)^2} = 1+\frac{1}{(x+1)^2}$ Since $(x+1)^2 \ge 0$, $\frac{1}{(x+1)^2} > 0$, then $\frac{dy}{dx} = 1+\frac{1}{(x+1)^2} > 1$. Since $\frac{dy}{dx} \ne 0$ for any real value of x, C has no stationary points.
	(ii) $y = \frac{x^2+5x+3}{x+1} = x+4-\frac{1}{x+1}$ Asymptotes: $y = x+4$, $x=-1$ y-intercept: when $x=0$, $y=3$ 

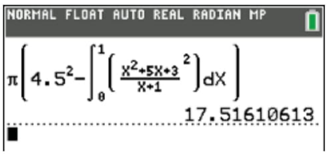
(iii)



Required volume $= \pi(4.5)^2(1) - \pi\int_0^1 \left(\dfrac{x^2+5x+3}{x+1}\right)^2 \mathrm{d}x$

$= 17.516$

$= 17.5$ units^3 (3 s.f.) (by G.C.)

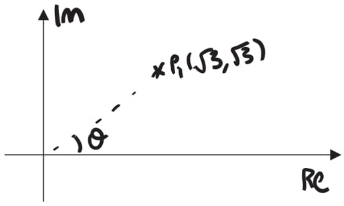


7 (i)

$|z| = \sqrt{3+3} = \sqrt{6}$

$\arg(z) = \frac{\pi}{4}$

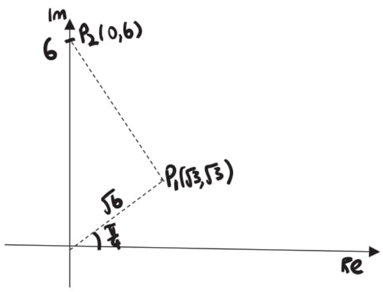
(refer to Argand diagram)



(ii)

$z = \sqrt{6}e^{i\left(\frac{\pi}{4}\right)}$

$\Rightarrow z^2 = 6e^{i\left(\frac{\pi}{2}\right)}$



Area of triangle $OP_1P_2 = \frac{1}{2}(6)\sqrt{3} = 3\sqrt{3}$

(iii)

$$\arg(w^n z^3) = \arg(w^n) + \arg(z^3)$$
$$= n\arg w + 3\arg z$$
$$= n\left(-\frac{\pi}{3}\right) + 3\left(\frac{\pi}{4}\right)$$

$\arg(w^n z^3) = -\frac{\pi}{4}$

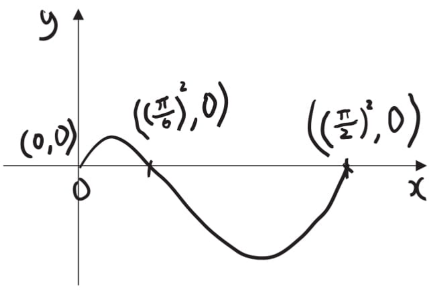
$\Rightarrow n\left(-\frac{\pi}{3}\right) + \frac{3\pi}{4} = -\frac{\pi}{4} + 2k\pi$ where $k \in \mathbb{Z}$

$$-\frac{n\pi}{3} = -\pi + 2k\pi$$
$$n = 3 - 6k$$

$\{n \in \mathbb{Z} : n = 3 - 6k, k \in \mathbb{Z}\}$

i.e. $n \in \{..., -9, -3, 3, 9, ...\}$

8	(a) $2^y = 2+\sin 2x$ Differentiate w.r.t x: $2^y \ln 2\frac{dy}{dx} = 2\cos 2x$ …(1) Differentiate w.r.t x: $2^y \ln 2\frac{d^2y}{dx^2} + 2^y(\ln 2)^2\left(\frac{dy}{dx}\right)^2 = -4\sin 2x$ …(2) When $x=0, y=1, \frac{dy}{dx} = \frac{1}{\ln 2}, \frac{d^2y}{dx^2} = -\frac{1}{\ln 2}$ $y = 1+\frac{1}{\ln 2}x - \frac{1}{2\ln 2}x^2 + ...$	Alternative method $y\ln 2 = \ln(2+\sin 2x)$ $(\ln 2)\frac{dy}{dx} = \frac{2\cos 2x}{2+\sin 2x}$ $(\ln 2)\frac{d^2y}{dx^2} = \frac{-4\sin 2x(2+\sin 2x)-(2\cos 2x)^2}{(2+\sin 2x)^2}$ When $x=0, y=1, \frac{dy}{dx} = \frac{1}{\ln 2}, \frac{d^2y}{dx^2} = -\frac{1}{\ln 2}$ $y = 1+\frac{1}{\ln 2}x - \frac{1}{2\ln 2}x^2 + ...$
	(b)(i) $BC = AC\cos\left(\frac{\pi}{3}\right)$, $DC = AC\sin\left(\frac{\pi}{6}-\theta\right)$ $\frac{BC}{DC} = \frac{AC\cos\left(\frac{\pi}{3}\right)}{AC\sin\left(\frac{\pi}{6}-\theta\right)} = \frac{1}{2\left(\sin\frac{\pi}{6}\cos\theta - \cos\frac{\pi}{6}\sin\theta\right)}$ $= \frac{1}{2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)}$ $= \frac{1}{\cos\theta - \sqrt{3}\sin\theta}$ (shown)	
	(b)(ii) Since θ is sufficiently small, $\frac{BC}{DC} \approx \frac{1}{1-\frac{\theta^2}{2}-\sqrt{3}\theta} = \left(1-\left(\sqrt{3}\theta+\frac{\theta^2}{2}\right)\right)^{-1}$ $= 1+\sqrt{3}\theta+\frac{\theta^2}{2}+\left(\sqrt{3}\theta+\frac{\theta^2}{2}\right)^2 + ...$ $= 1+\sqrt{3}\theta+\frac{\theta^2}{2}+3\theta^2+...$ $\approx 1+\sqrt{3}\theta+\frac{7\theta^2}{2}$ $\therefore a=\sqrt{3}, b=\frac{7}{2}$	

9	(a)(i) Using factor formula (MF26), $2\sin x\cos 3x = \sin 4x - \sin 2x$. Hence $\int 2\sin x\cos 3x\,\mathrm{d}x = \int(\sin 4x - \sin 2x)\,\mathrm{d}x$ $= -\frac{\cos 4x}{4} + \frac{\cos 2x}{2} + C$
	(a)(ii) Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ $\frac{dv}{dx} = 2\sin x\cos x \Rightarrow v = -\frac{\cos 4x}{4} + \frac{\cos 2x}{2}$ $\int 2x\sin x\cos 3x\,\mathrm{d}x$ $= x\left(-\frac{\cos 4x}{4} + \frac{\cos 2x}{2}\right) - \int -\frac{\cos 4x}{4} + \frac{\cos 2x}{2}\,\mathrm{d}x$ $= -\frac{x\cos 4x}{4} + \frac{x\cos 2x}{2} + \frac{\sin 4x}{16} - \frac{\sin 2x}{4} + C$ $= \frac{1}{16}[-4x\cos 4x + 8x\cos 2x + \sin 4x - 4\sin 2x] + C$
	(b)(i)  To find x-intercepts, $y = \sin\theta\cos 3\theta = 0$ $\sin\theta = 0$ or $\cos 3\theta = 0$ $\theta = 0$ or $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $(0 \le \theta \le \frac{\pi}{2} \Rightarrow 0 \le 3\theta \le \frac{3\pi}{2})$ $\theta = \frac{\pi}{6}, \frac{\pi}{2}$ Thus intercepts are when $\theta = 0, \frac{\pi}{6}, \frac{\pi}{2}$.

	(b)(ii) Area of S $=\int_0^{\left(\frac{\pi}{6}\right)^2} y\,dx-\int_{\left(\frac{\pi}{6}\right)^2}^{\left(\frac{\pi}{2}\right)^2} y\,dx$ $=\int_0^{\frac{\pi}{6}}\sin\theta\cos3\theta(2\theta)\,d\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin\theta\cos3\theta(2\theta)\,d\theta$ $=\int_0^{\frac{\pi}{6}}2\theta\sin\theta\cos3\theta\,d\theta-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}2\theta\sin\theta\cos3\theta\,d\theta$ $=\frac{1}{16}\left[-4\theta\cos4\theta+8\theta\cos2\theta+\sin4\theta-4\sin2\theta\right]_0^{\frac{\pi}{6}}$ $-\frac{1}{16}\left[-4\theta\cos4\theta+8\theta\cos2\theta+\sin4\theta-4\sin2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $=\frac{1}{16}\left(-4\left(\frac{\pi}{6}\right)\left(-\frac{1}{2}\right)+8\left(\frac{\pi}{6}\right)\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}-4\left(\frac{\sqrt{3}}{2}\right)\right)-\frac{1}{16}(0)$ $-\frac{1}{16}\left(-2\pi+4\pi(-1)\right)+\frac{1}{16}\left(-4\left(\frac{\pi}{6}\right)\left(-\frac{1}{2}\right)+8\left(\frac{\pi}{6}\right)\left(\frac{1}{2}\right)+\frac{\sqrt{3}}{2}-4\left(\frac{\sqrt{3}}{2}\right)\right)$ $=2\times\frac{1}{16}\left(\frac{\pi}{3}+\frac{2\pi}{3}+\frac{\sqrt{3}}{2}-2\sqrt{3}\right)-\frac{1}{16}\left[-6\pi\right]$ $=\frac{\pi}{2}-\frac{3}{16}\sqrt{3}$

10	(i) Since $\frac{dT}{dt}>0$ as the object is being heated up, and $T_H-T>0$ as hotplate temperature is higher than that of the object, it follows that k is positive.
	(ii) $\frac{dT}{dt}=k(275-T)$ $\int\frac{1}{275-T}\,dT=\int k\,dt$ $-\ln(275-T)=kt+C$ $275-T=Ae^{-kt}$ where $A=e^{-C}$ Substituting $t=0$, $T=25$, $250=Ae^0$ thus $A=250$ $T=275-250e^{-kt}$ Substituting $t=100$, $T=75$, $75=275-250e^{-100k}$ $k\approx0.0022314$ So $T=275-250e^{-0.0022314t}$.

(iii)
Curve B is a possible graph. Curve A does not fit because:

- Temperature does not exceed equilibrium as object is being heated continuously;

OR

- The curve cannot have different gradients for same value of T *(note that the* $\frac{\mathrm{d}T}{\mathrm{d}t}$ *is linear in* T*)*;

OR

- Gradient cannot be negative at any point because the object is being heated continuously.

OR

- Observe that

$$\frac{\mathrm{d}T}{\mathrm{d}t}=k\left(T_H-T\right)-m\left(T-T_S\right)$$
$$=\left(k+m\right)\left(\frac{kT_H+mT_S}{k+m}-T\right)$$

So $\frac{\mathrm{d}T}{\mathrm{d}t}$ is always > 0.

(iv)
As $T\to125$, $\frac{\mathrm{d}T}{\mathrm{d}t}\to k(275-125)-m(125-25)$.
From graph,
as $T\to125$, $\frac{\mathrm{d}T}{\mathrm{d}t}\to0$.
So, $0=k(275-125)-m(125-25)$
$\Rightarrow m=\frac{3k}{2}\approx0.00335$ (3s.f.)

11 (i)

$$\overrightarrow{PQ}=\begin{pmatrix}9\\8\\3\end{pmatrix}-\begin{pmatrix}9\\2\\9\end{pmatrix}=\begin{pmatrix}0\\6\\-6\end{pmatrix},\ \overrightarrow{PR}=\begin{pmatrix}3\\2\\3\end{pmatrix}-\begin{pmatrix}9\\2\\9\end{pmatrix}=\begin{pmatrix}-6\\0\\-6\end{pmatrix}$$

A vector normal to Π_1 is $\begin{pmatrix}0\\1\\-1\end{pmatrix}\times\begin{pmatrix}-1\\0\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$

$$\mathbf{r}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=2$$

So a Cartesian equation of Π_1 is $-x+y+z=2$

(ii)
Position vector of midpoint of PQ is

$$\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OQ}\right)=\begin{pmatrix}9\\5\\6\end{pmatrix}$$

Π_2 is perpendicular to $\overrightarrow{PQ}$, so $\overrightarrow{PQ}$ is normal to Π_2

So $\mathbf{r}.\begin{pmatrix}0\\6\\-6\end{pmatrix}=\begin{pmatrix}9\\5\\6\end{pmatrix}.\begin{pmatrix}0\\6\\-6\end{pmatrix}=-6$

So a Cartesian equation of Π_2 is $6y-6z=-6 \Rightarrow y-z=-1$.

(iii)

Eqn of line passing through S and F is

$\mathbf{r}=\begin{pmatrix}3\\8\\9\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix},\quad \lambda\in\mathbb{R}$

So $\overrightarrow{OF}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}$ for some λ

F lies on Π_1

So $\overrightarrow{OF}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=2$

$\Rightarrow \lambda=-4$

So coordinates of F are $(7,4,5)$.

(iv)

Note that T lies on the line SF,

So $\overrightarrow{OT}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}$ for some λ from (iii)

$\overrightarrow{PT}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}-\begin{pmatrix}9\\2\\9\end{pmatrix}=\begin{pmatrix}-6-\lambda\\6+\lambda\\\lambda\end{pmatrix}$ and

$\overrightarrow{ST}=\begin{pmatrix}3-\lambda\\8+\lambda\\9+\lambda\end{pmatrix}-\begin{pmatrix}3\\8\\9\end{pmatrix}=\begin{pmatrix}-\lambda\\\lambda\\\lambda\end{pmatrix}$

Since $\left|\overrightarrow{PT}\right|=\left|\overrightarrow{ST}\right|$,

$(6+\lambda)^2+(6+\lambda)^2+\lambda^2=(-\lambda)^2+\lambda^2+\lambda^2$

$\Rightarrow (6+\lambda)^2=\lambda^2$

$\Rightarrow (6+\lambda)^2-\lambda^2=0$

$\Rightarrow (6+\lambda+\lambda)(6+\lambda-\lambda)=0$

$\Rightarrow \lambda=-3$

Hence coordinates of $T(6,5,6)$.

End of Paper

EJC_H2_2019_JC2_Prelim_P2_Solutions

Section A: Pure Mathematics [40 marks]

1	(a) Sub $z=(2+2\text{i})w$ into the other equation $\Rightarrow(1-2\text{i})(2+2\text{i})w=39-11w\text{i}$ $\Rightarrow w=\dfrac{39}{(1-2\text{i})(2+2\text{i})+11\text{i}}=2-3\text{i}$ (using GC) Thus, $z=(2+2\text{i})(2-3\text{i})=10-2\text{i}$	OR Sub $w=\dfrac{z}{2+2\text{i}}$ into the other equation $\Rightarrow(1-2\text{i})z=39-11\text{i}\left(\dfrac{z}{2+2\text{i}}\right)$ $\Rightarrow z=\dfrac{39}{\dfrac{11\text{i}}{2+2\text{i}}+(1-2\text{i})}=10-2\text{i}$ (using GC) Thus, $w=\dfrac{10-2\text{i}}{2+2\text{i}}=2-3\text{i}$.
	(b) $(1+\text{i}c)^3=1+3\text{i}c+3(\text{i}c)^2+(\text{i}c)^3$ $=1+3\text{i}c-3c^2-\text{i}c^3$ $=1-3c^2+\text{i}(3c-c^3)$ Since $(1+\text{i}c)^3$ is real, $3c-c^3=0$ $c(3-c^2)=0$ $c=0,\pm\sqrt{3}$	

2

(i)

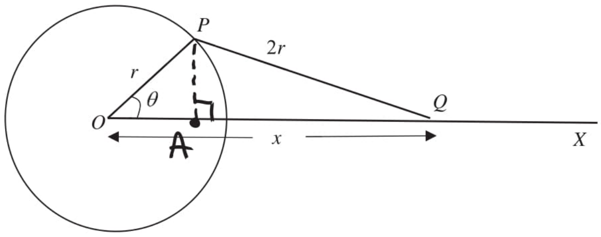
Max $x = 3r$ when $\theta = 0$

Min $x = r$ when $\theta = \pi$

(ii)

Method 1

Consider triangle OPA.



$$\cos\theta = \frac{OA}{r} \Rightarrow OA = r\cos\theta$$

Consider triangle PAQ. By pythagoras theorem,

$$AQ = \sqrt{(2r)^2 - (PA)^2}$$
$$= \sqrt{(2r)^2 - (r\sin\theta)^2}$$
$$= r\sqrt{4 - \sin^2\theta}$$

$$x = OA + AQ = r\cos\theta + r\sqrt{4-\sin^2\theta} = r\left[\cos\theta + \sqrt{4-\sin^2\theta}\right] \text{ (shown)}$$

Method 2: Cosine Rule

$$(2r)^2 = r^2 + x^2 - 2rx\cos\theta$$
$$4r^2 = r^2 + x^2 - 2rx\cos\theta$$
$$= (x - r\cos\theta)^2 + r^2\sin^2\theta$$
$$x - r\cos\theta = r\sqrt{4-\sin^2\theta} \quad \left(\text{reject } -r\sqrt{4-\sin^2\theta} \because x \ge r \ge r\cos\theta\right)$$
$$x = r\left(\cos\theta + \sqrt{4-\sin^2\theta}\right) \quad \text{(shown)}$$

(iii)

Method 1:

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$
$$= r\left[-\sin\theta + \frac{(-2\sin\theta\cos\theta)}{2\sqrt{4-\sin^2\theta}}\right] \times \frac{d\theta}{dt}$$

When $\theta = \frac{\pi}{6}$ and $\frac{d\theta}{dt} = 0.3$,

$$\frac{dx}{dt} = r\left[-\sin\frac{\pi}{6} - \frac{\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\sqrt{4-\sin^2\left(\frac{\pi}{6}\right)}}\right](0.3)$$
$$= -0.217r$$

Method 2:

Differentiate implicitly w.r.t t,

$$\frac{dx}{dt}=r\left(\sin\theta\frac{d\theta}{dt}+\frac{(-2\sin\theta\cos\theta)}{2\sqrt{4-\sin^2\theta}}\frac{d\theta}{dt}\right)$$

When $\theta=\frac{\pi}{6}$ and $\frac{d\theta}{dt}=0.3$,

$$\frac{dx}{dt}=r\left[-\left(\sin\frac{\pi}{6}\right)(0.3)-\frac{\sin\frac{\pi}{6}\cos\frac{\pi}{6}}{\sqrt{4-\sin^2\left(\frac{\pi}{6}\right)}}(0.3)\right]$$

$$=-0.217r$$

---

3 (i)

Length of projection of q onto p $=|\mathbf{q}.\hat{\mathbf{p}}|=\left|\frac{\mathbf{q}.\mathbf{p}}{|\mathbf{p}|}\right|$

Method 1

$3\overrightarrow{PR}=5\overrightarrow{PQ}\Rightarrow 3(\mathbf{r}-\mathbf{p})=5(\mathbf{q}-\mathbf{p})\Rightarrow \mathbf{q}=\frac{1}{5}(2\mathbf{p}+3\mathbf{r})$

Sub into $|\mathbf{q}.\hat{\mathbf{p}}|$:

$$|\mathbf{q}.\hat{\mathbf{p}}|=\left|\frac{\frac{1}{5}(2\mathbf{p}+3\mathbf{r}).\mathbf{p}}{|\mathbf{p}|}\right|$$

$$=\left|\frac{\frac{2}{5}\mathbf{p}.\mathbf{p}+\frac{3}{5}\mathbf{p}.\mathbf{r}}{|\mathbf{p}|}\right|$$

$$=\frac{\frac{2}{5}(29)+\frac{3}{5}(11)}{\sqrt{29}}=\frac{91}{5\sqrt{29}}\text{ (or 3.38)}$$

Method 2

$3\overrightarrow{PR}=5\overrightarrow{PQ}\Rightarrow 3(\mathbf{r}-\mathbf{p})=5(\mathbf{q}-\mathbf{p})\Rightarrow \mathbf{r}=\frac{1}{3}(5\mathbf{q}-2\mathbf{p})$

Sub into $\mathbf{p}.\mathbf{r}=11$:

$\Rightarrow \frac{1}{3}(5\mathbf{q}-2\mathbf{p}).\mathbf{p}=11$

$\Rightarrow 5\mathbf{q}.\mathbf{p}-2\mathbf{p}.\mathbf{p}=33$

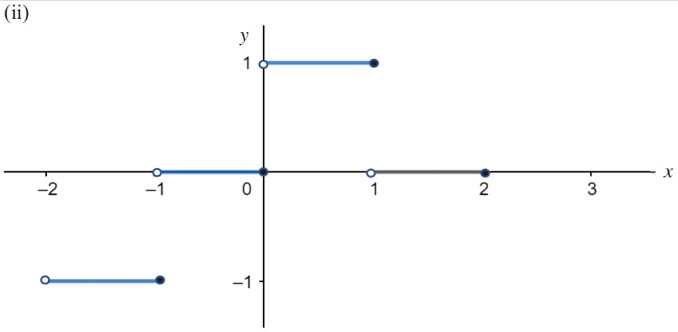
$\Rightarrow \mathbf{q}.\mathbf{p}=\frac{91}{5}\Rightarrow |\mathbf{q}.\hat{\mathbf{p}}|=\frac{91}{5\sqrt{29}}$ (or 3.38)

Method 3

Dot p to both sides,

$3(\mathbf{r}-\mathbf{p}).\mathbf{p}=5(\mathbf{q}-\mathbf{p}).\mathbf{p}$

	$\Rightarrow 3(\mathbf{r}-\mathbf{p}).\mathbf{p}=5(\mathbf{q}-\mathbf{p}).\mathbf{p}$ $\Rightarrow 3\mathbf{r.p}-3\mathbf{p.p}=5\mathbf{q.p}-5\mathbf{p.p}$ $\Rightarrow \mathbf{p.q}=\frac{1}{5}(3\mathbf{p.r}+2\mathbf{p.p})=\frac{1}{5}\left(3(11)+2\left(\sqrt{29}\right)^2\right)=\frac{91}{5}$ So $\lvert\mathbf{q.\hat{p}}\rvert=\frac{91}{5\sqrt{29}}$ (or 3.38)
	(ii) $\overrightarrow{PS}=\mathbf{r}$ so *OPSR* is a parallelogram spanned by *OP* and *OR*. So area of *OPSR* $=\lvert\mathbf{p}\times\mathbf{r}\rvert$ $=\left\lvert\begin{pmatrix}3\\2\\-4\end{pmatrix}\times\begin{pmatrix}1\\-2\\-3\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}-6-8\\-(-9+4)\\-6-2\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}-14\\5\\-8\end{pmatrix}\right\rvert=\sqrt{285}$

4	(i) $\mathrm{f}(-1.4)=\lceil -1.4\rceil=-1$
	(ii) 
	(iii) Method 1 No, because the horizontal line $y=1$ (for example) cuts the graph more than once from $(0,1]$. So f is not 1-1 so f^{-1} does not exist. Method 2 No, because for example, $\mathrm{f}(1.1)=\mathrm{f}(1.2)=0$. So f is not 1-1 so f^{-1} does not exist.
	(iv) $R_{\mathrm{f}}=\{-1,0,1\}$
	(v)

	$g^2(x)=\dfrac{a\left(\dfrac{ax-3}{x-a}\right)-3}{\dfrac{ax-3}{x-a}-a}$ $=\dfrac{a^2x-3a-3x+3a}{ax-3-ax+a^2}$ $=x$ Then $g^3(x)=g^2(g(x))$ $=\dfrac{ax-3}{x-a}$ Observe that even compositions give x, odd compositions give g(x). So $g^{2019}(x)=\dfrac{ax-3}{x-a}\Rightarrow g^{2019}(5)=\dfrac{5a-3}{5-a}$.
	(vi) $g(x)=\dfrac{3x-3}{x-3}$ $D_f=(-2,2]\xrightarrow{f}\{-1,0,1\}\xrightarrow{g}\left\{\dfrac{3}{2},1,0\right\}$

5	(a)(i) $u_1=\dfrac{4}{M^2},u_2=\dfrac{4}{M^5},u_3=\dfrac{4}{M^8}$
	(a) (ii) $\displaystyle\sum_{r=1}^{n}\frac{4}{M^{3r-1}}=\frac{\frac{4}{M^2}\left(1-\frac{1}{M^{3n}}\right)}{1-\frac{1}{M^3}}$ $=\dfrac{4}{M^2}\left(1-\dfrac{1}{M^{3n}}\right)\times\dfrac{M^3}{M^3-1}$ $=\dfrac{4M}{M^3-1}\left(1-\dfrac{1}{M^{3n}}\right)$ (shown)
	(a) (iii) Method 1 (consider expression) Since as $n\to\infty$, $\dfrac{1}{M^{3n}}\to 0$, $\therefore\dfrac{4M}{M^3-1}\left(1-\dfrac{1}{M^{3n}}\right)\to\dfrac{4M}{M^3-1}$ The sum to infinity is $\dfrac{4M}{M^3-1}$. Method 2 (consider GP) This is a GP with common ratio = $\dfrac{1}{M^3}$

	$M>1 \Rightarrow 0<\frac{1}{M}<1 \Rightarrow 0<\frac{1}{M^3}<1$, so the series is convergent. $S=\dfrac{\frac{4}{M^2}}{1-\frac{1}{M^3}}=\dfrac{4M}{M^3-1}$
	(b)(i) $\cos\left(\frac{2r+1}{2}\right)-\cos\left(\frac{2r-1}{2}\right)$ $=-2\sin\left(\frac{1}{2}\left(\frac{2r+1}{2}+\frac{2r-1}{2}\right)\right)\sin\left(\frac{1}{2}\left(\frac{2r+1}{2}-\frac{2r-1}{2}\right)\right)$ $=-2\sin(r)\sin\left(\frac{1}{2}\right)$ (shown)
	(b)(ii) $\sum_{r=1}^{n}\sin r=-\frac{1}{2\sin\left(\frac{1}{2}\right)}\sum_{r=1}^{n}\left(\cos\left(\frac{2r+1}{2}\right)-\cos\left(\frac{2r-1}{2}\right)\right)$ $=-\frac{1}{2\sin\left(\frac{1}{2}\right)}\begin{bmatrix}\cos\left(\frac{3}{2}\right)-\cos\left(\frac{1}{2}\right)\\+\cos\left(\frac{5}{2}\right)-\cos\left(\frac{3}{2}\right)\\+\cos\left(\frac{7}{2}\right)-\cos\left(\frac{5}{2}\right)\\+\quad\vdots\\+\cos\left(\frac{2n-1}{2}\right)-\cos\left(\frac{2n-3}{2}\right)\\+\cos\left(\frac{2n+1}{2}\right)-\cos\left(\frac{2n-1}{2}\right)\end{bmatrix}$ $=-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left[\cos\left(n+\frac{1}{2}\right)-\cos\left(\frac{1}{2}\right)\right]$ $=\operatorname{cosec}\left(\frac{1}{2}\right)\sin\frac{n+1}{2}\sin\frac{n}{2}$ shown

Section B: Probability and Statistics [60 marks]

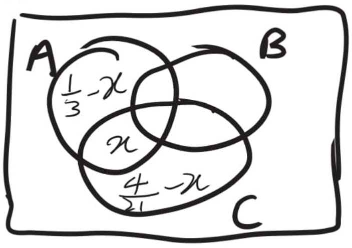
6	(i) $d = 0.5 - c$
	(ii) Let X be the result of one throw of the die. $\text{E}(X) = (1)(0.3) + (2)(c) + (3)(0.5 - c) + (4)(0.2) = 2.6 - c$ $\text{E}(X^2) = (1)(0.3) + (4)(c) + (9)(0.5 - c) + (16)(0.2) = 8 - 5c$ $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$ $= (8 - 5c) - (2.6 - c)^2$ $= 8 - 5c - c^2 + 5.2c - 6.76$ $= -c^2 + 0.2c + 1.24$ $= -(c - 0.1)^2 + 1.25$ (completing the square) Thus, the variance is maximum when $c = 0.1$
	(iii) Let Y be the number of throws, out of 10, that land on an even number. $Y \sim \text{B}(10, 0.4)$ Required probability $= \text{P}(Y \geq 7)$ $= 1 - \text{P}(Y \leq 6)$ $= 0.054762...$ $= 0.0548$ (to 3 s.f.)

7	(i) $X_2 \sim \text{N}(\mu, 4)$ $\text{P}(\mu - 1 < X_2 < \mu + 1)$ $= \text{P}\left(\frac{\mu - 1 - \mu}{2} < \frac{X_2 - \mu}{2} < \frac{\mu + 1 - \mu}{2}\right)$ $= \text{P}\left(-\frac{1}{2} < Z < \frac{1}{2}\right)$ where $Z \sim \text{N}(0, 1)$ $= 0.38292\ldots$ ≈ 0.383 (3 s.f.)
	(ii) $X_3 - X_4 \sim \text{N}(0, 14)$ $\text{P}(X_3 \geq X_4) = \text{P}(X_3 - X_4 \geq 0) = \frac{1}{2}$ (by symmetry)
	(iii) $\text{Var}(Y_n) = \frac{1}{n^2}\text{Var}(X_1 + X_2 + \ldots + X_n)$ $= \frac{1}{n^2}\left(\text{Var}(X_1) + \text{Var}(X_2) + \ldots + \text{Var}(X_n)\right)$ $= \frac{1}{n^2}\left(2 + 4 + 6 + \ldots + 2n\right)$

	$=\frac{1}{n^2}\times n(n+1)$ (Sum of A.P.) $=1+\frac{1}{n}$ Since the X_n's are independent Normal distributions with common mean, $Y_n \sim \mathrm{N}\left(\mu, 1+\frac{1}{n}\right)$ (NB: The variance of Y_n decreases as n increases.) Either $\mathrm{P}(\mu-1<Y_n<\mu+1)>\frac{2}{3}$ $\mathrm{P}\left(\frac{\mu-1-\mu}{\sqrt{1+\frac{1}{n}}}<\frac{Y_n-\mu}{\sqrt{1+\frac{1}{n}}}<\frac{\mu+1-\mu}{\sqrt{1+\frac{1}{n}}}\right)>\frac{2}{3}$ $\mathrm{P}\left(\frac{-1}{\sqrt{1+\frac{1}{n}}}<Z<\frac{1}{\sqrt{1+\frac{1}{n}}}\right)>\frac{2}{3}$ $\frac{1}{\sqrt{1+\frac{1}{n}}}>0.96742$ Solving this inequality, $n>14.6017\ldots$ Hence, the smallest possible value of n is 15. Alternatively $Y_n-\mu \sim \mathrm{N}\left(0, 1+\frac{1}{n}\right)$ From GC, n | $\mathrm{P}(-1<Y_n-\mu<1)$ 14 | 0.6660 15 | 0.6671 $\therefore$ smallest value of n is 15.

8	(i) $\mathrm{P}(B)=\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B')=\frac{6}{7}-\frac{1}{3}=\frac{11}{21}$
	(ii) $\mathrm{P}(A' \mid B)=\frac{\mathrm{P}(A'\cap B)}{\mathrm{P}(B)}=\frac{\mathrm{P}(A\cup B)-\mathrm{P}(A)}{\mathrm{P}(B)}=\frac{\frac{6}{7}-\frac{2}{5}}{\frac{11}{21}}=\frac{\frac{16}{35}}{\frac{11}{21}}=\frac{48}{55}$
	(iii) $\mathrm{P}(B'\cap C)=P(C)-P(B\cap C)$ $=\frac{2}{5}-P(B)P(C)$ ($\because B, C$ independent) $=\frac{2}{5}-\frac{11}{21}\times\frac{2}{5}$ $=\frac{4}{21}$

(iv)



Let $\mathrm{P}(A \cap B' \cap C) = x$

Since $\frac{4}{21} - x \geq 0$, $x \leq \frac{4}{21}$

Furthermore, since $\mathrm{P}(A \cup B) = \frac{6}{7}$, $\frac{4}{21} - x \leq \frac{1}{7}$, so $x \geq \frac{1}{21}$

<u>Alternative:</u>

$A \cap B' \cap C \subseteq B' \cap C \Rightarrow P(A \cap B' \cap C) \leq P(B' \cap C)$

So greatest possible value is $\frac{4}{21}$.

Furthermore, $P(A \cap B' \cap C) = P(B' \cap C) - P(A' \cap B' \cap C)$

And $P(A' \cap B' \cap C) \leq P(A' \cap B') = 1 - P(A \cup B) = \frac{1}{7}$

So $P(A \cap B' \cap C) \geq P(B' \cap C) - \frac{1}{7} = \frac{1}{21}$

So least possible value is $\frac{1}{21}$.

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9 (a)(i)

9 letters with 3 'E' and 2 'L'

No. of ways $= \frac{9!}{3!2!} = 30240$

(a)(ii)

L is fixed

_J_W_L_R_Y_: $5! = 120$

Case 1: separated by 2 and 2 – 2 ways

Case 2: separated by 2 and 3 / 3 and 2 – 2 ways

Total number of ways: $120 \times (2 + 2) = 480$ ways

(a)(iii)

All distinct: ${}^6C_4 \times 4! = 360$

EE or LL (but not both): ${}^2C_1 \times {}^5C_2 \times \frac{4!}{2!} = 240$

EE and LL: $\frac{4!}{2!2!} = 6$

EEE: ${}^5C_1 \times \frac{4!}{3!} = 20$

Total: $360 + 240 + 6 + 20 = 626$

(b)

Mr and Mrs Lee together: $(9-1)!\ \times 2! = 80640$

Mr and Mrs Lee together and 3 children together: $(7-1)!\ \times 2!\ \times 3! = 8640$

Number of ways: $80640 - 8640 = 72000$

OR

Let A be the event that Mr and Mrs Lee are seated together and
B be the event that the 3 children are all seated together.

Then no. of ways $= n(A) - n(A \cap B)$

$= (9-1)!\ \times 2!\ - (7-1)!\ \times 2!\ \times 3!$

$= 80640 - 8640 = 72000$

10 (i)

$\bar{x} = 1050 + \frac{58.0}{50} = 1051.16$

$s^2 = \frac{1}{n-1}\left(\sum(x-1050)^2 - \frac{\left[\sum(x-1050)\right]^2}{n}\right)$

$= \frac{1}{49}\left(2326 - \frac{58.0^2}{50}\right)$

$= 46.096$ (5 s.f.)

To test H$_0$: $\mu = 1053$ against

H$_1$: $\mu \neq 1053$ at 5% level of significance

Since $n = 50$ is large, by Central Limit Theorem,

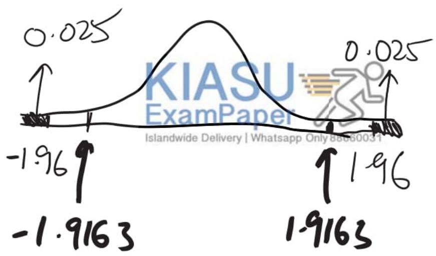
under H$_0$, $\bar{X} \sim \mathrm{N}\left(1053, \frac{46.096}{50}\right)$ approximately

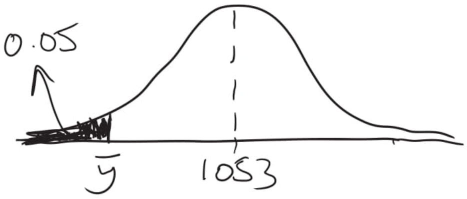
either

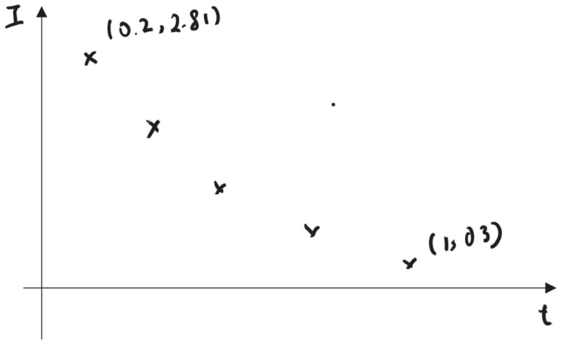
p-value: 0.055322

or

z-value: -1.9163 and critical region: $|z_{0.025}| = 1.96$



	Since p-value > 0.05 (or |z-value| < 1.96), we do not reject H_0 and conclude at 5% level of significance that there is insufficient evidence that the population mean amount of sodium per packet has changed after alterations to the workflow.
	(ii) The probability of wrongly concluding that the mean amount of sodium is not 1053mg, when it is in fact 1053mg, is 0.05.
	(iii) Since p-value is 0.055322, $\alpha \geq 6$
	(iv) If we tested H_1: $\mu < 1053$, Either $p\text{-value} = 0.027661 < 0.05$ or $z\text{-value} = -1.9163 < -1.645$ So we may reject H_0 and conclude at 5% level of significance that the population mean amount of sodium had decreased.
	(v) To test H_0: $\mu = 1053$ against H_1: $\mu < 1053$ at 5% level of significance Since $n = 50$ is large, by Central Limit Theorem, under H_0, $\bar{X} \sim N\left(1053, \frac{6.0^2}{40}\right)$ approximately To reject H_0, p-value < 0.05  $\Rightarrow \bar{y} < 1051.4$ (to 1 d.p.) It is not necessary to assume anything about the population distribution, as sample size (= 40) is large enough, so the Central Limit Theorem says the sample mean amount of sodium approximately follows a normal distribution.

11	(a) There may be a strong negative linear correlation between the amount of red wine intake and the risk of heart disease, but we cannot conclude that amount of red wine intake causes risk of heart disease to decrease, as causality cannot be inferred from correlation.
	(b)(i) The variable t is the independent variable, as we are able to control, or determine, the intervals at which we measure the corresponding radiation.
	(b)(ii)  From the scatter diagram, we can see that the points lie along a curve, rather than a straight line. Hence $I = at + b$ is not a likely model.
	(b)(iii) r between I and t = -0.9565 r between ln I and t = -0.9998
	(b)(iv) $I = ae^{bt} \Rightarrow \ln I = bt + \ln a$ Equation of regression line: $\ln I = -2.7834239t + 1.6007544 \Rightarrow \ln I = -2.78t + 1.60$ $\ln a = 1.600754 \Rightarrow a = 4.96$ (3 s.f.) $b = -2.78$ (3 s.f.)
	(b)(v) $t = 0.7$, $I = 0.706$ (to 3 sig fig) The answer is reliable as r is close to -1, and $t = 0.7$ is within the data range (0.2 to 1.0) and thus the estimate is obtained via interpolation.

End of Paper